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MODELING OF SYSTEMS USING ITÔ'S STOCHASTIC DIFFERENTIAL EQUATIONS

MODELADO DE SISTEMAS USANDO ECUACIONES DIFERENCIALES ESTOCÁSTICAS DE ITÔ

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Abstract

This paper deals with the modeling of systems subject to random perturbations. The main objective is to compare the experimentally measured trajectories with the solutions of the ordinary differential equation (ODE) and the stochastic differential equation (SDE) which model the systems analyzed with the purpose of verify if the SDEs capture the random perturbations and therefore, are more appropriate to describe the phenomena with random noise. To this end, the Itô's calculus is used and numerical simulations of the SDEs are done in MATLAB using the Euler-Maruyama method. As an application of the SDEs, an optimal investment problem is solved in analytic form by following the standard dynamic programming technique.

Keywords: Brownian motion, Itô's calculus, optimal control, stochastic processes, white noise.

Resumen

El trabajo estudia el modelado de sistemas que presentan perturbaciones aleatorias. El principal objetivo consiste en comparar las trayectorias obtenidas de las mediciones experimentales con las trayectorias de la ecuación diferencial ordinaria y la ecuación diferencial estocástica (EDE) que modelan los sistemas analizados con el propósito de verificar si las EDEs capturan las pertubaciones aleatorias y por lo tanto, son más apropiadas para describir los fenómenos con ruido aleatorio. Para este fin, se usa el cálculo de Itô y las simulaciones numéricas se hacen en MATLAB usando el método de Euler-Maruyama. Como una aplicación de las EDEs, se resuelve de manera analítica un problema de inversión óptima usando la técina de programación dinámica.

Palabras clave: movimiento browniano, cálculo de Itô, control óptimo, procesos estocásticos, ruido blanco.

1 Introduction

In many applications, the experimentally measured trajectories of the systems modeled are apparently subject to random perturbations, the observed state seems to more or less follow the trajectory predicted by an ordinary differential equation (ODE), but not exactly. The SDEs arises when the random effects disturbing the system are considered in the modeling of the phenomenona. The aim of this work to compare the experimentally measured trajectories with the solution of the SDE and ODE of the systems modeled. To this end, applications in which it was possible to obtain experimental data are studied. The Itô's calculus allowed us to find the analytic solutions of the SDEs. Moreover, as an application of the SDEs an optimal investment problem is solved using the dynamic programming approach.

Ordinary differential equations are used to describe

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the evolution of systems or processes in the nature, but much of this process involves random fluctuations (known noise) which the ODEs do not consider. Hence, it seems reasonable to modify the ODEs, somehow to include the possibility of random effects disturbing the system. This manner, stochastic differential equations (SDEs) arise when a random white noise is introduced into ODEs. This random white noise can be thought of as the derivative of Brownian motion (or the Wiener process) (see CKlebaner (2005) for more details). To illustrate this fact, consider the single-species deterministic population dynamic model

$$dN(t) = a(t)N(t)dt, \;\; N(0) = N_0, \;\; t \geq 0, \qquad (1)$$

where $N(t)$ denotes the population size and $a(t)$ is the population growth rate. Under some standard assumptions, there exists a unique solution $N(t)$ to ODE (1). Nevertheless, given that populations systems are often subject to environmental noise, it is important to discover whether the presence of the such noise affects the solution $N(t)$. Suppose that the population growth rate is constant $a(t) = r$ and moreover is stochastically perturbed, i.e.,

$$a(t) = r + \alpha\xi(t), \;\; r > 0, \qquad (2)$$

where $\xi(t)$ is white noise and $\alpha > 0$ represent the intensity of the noise. Then replacing stochastic population growth rate (2) in the deterministic population dynamic (1), it obtains the stochastic differential equation

$$dN(t) = rN(t)dt + \alpha N(t)\xi(t)dt, \;\; t \geq 0. \qquad (3)$$

This SDE is known as Itô's SDE in honor to the mathematician Kiyoshi Itô, since developed the *Itô's stochastic calculus theory*. The basic concept of this theory is the *Itô's integral*, and the most important of the results is the *Itô's Lemma*. Stochastic calculus became a very useful tool for applied problems and it allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to stochastic processes. With Itô's stochastic calculus it possible to obtain a solution to the SDE (3), which is given by

$$N(t) = N(0)\exp\left[(r - \frac{\alpha^2}{2}) + \alpha W(t)\right], \;\; t \geq 0,$$

where $W(t)$ is a Brownian motion.

Stochastic differential equations have been studied in, for instance, Durret (1996), Evans (2013), Friedman (2007), Karatzas and Shereve (1998), and Morimoto (2010). The systems that evolves according a SDEs in different areas are, the labor supply, the price of stocks, or the price of capital at time $t \geq 0$ in economic applications, whereas, the evolution of population growth and genetic evolution in biology, in engineering and in physics, the signals contaminated by a noise and the random oscillators with white noise, respectively. Also, in stochastic optimal control and filtering problems, Itô's SDEs are employed as models of dynamical systems disturbed by noise. See CKlebaner (2005), for details of the stochastic calculus used in the study of this systems.

Stochastic optimal control theory is a subfield of the control theory which seek to optimize a cost/reward functional over certain control functions and where the systems evolves according a stochastic differential equation controlled. An optimal control can be derived using Pontryagin's maximum principle or solving the Hamilton-Jacobi-Bellman equation (dynamic programming approach), Friedman (2007).

The structure of this work is as follows. Section 2 introduces the notion of SDEs and the Ito's calculus. Section 3 show some applications which allow us to compare the experimental data together the stochastic and deterministic solutions from SDE and ODE, respectively. On the other hand, Section 4 it describes briefly the stochastic optimal control (OSC) problem and the OSC theory is applied to an optimal investment problem. Finally, Section 5 states some concluding remarks.

2 Stochastic differential equations: Itô's stochastic calculus

Since there are excellent books which give a detailed description of the Itô's stochastic calculus, in this section we give a brief account of this theory and recommended the reader to these books for more information, CKlebaner (2005), Durret (1996), Evans (2013), Friedman (2007), Karatzas and Shereve (1991),(1998).

Definition 2.1 *(White noise process.) A* white noise process, $\xi(t)$, *is a stochastic process such that*

(a) $\xi(t)$ *is a Gaussian random variable with zero mean and variance 1, i.e.,* $\mathbb{E}[\xi(t)] = 0$ *and* $\mathbb{E}[\xi^2(t)] = 1$.

(b) *$\xi(t)$ and $\xi(s)$ are not correlated if $t \neq s$, i.e., the covariance function*

$$C(t) = \mathbb{E}[\xi(t+s)\xi(s)]$$

is the Dirac delta

$$C(t) = \delta(t) = \begin{cases} 0 & \text{si } t \neq 0 \\ 1 & \text{si } t = 0 \end{cases} \tag{4}$$

Remark 2.2 *The Fourier transform of the covariance function $C(t)$, also known as spectral density of the process $\xi(t)$ is constant:*

$$h(\lambda) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}C(t)dt = \frac{1}{2\pi}, \quad -\infty < \lambda < \infty. \tag{5}$$

That is, the spectrum of the process $\xi(t)$ is white *because all frequencies appear with the same intensity (in analogy with the "white" light that uniformly contains all the frequencies of visible light).*

From (5), the covariance function $C(t)$ can be obtained by the inverse transform

$$C(t) = \int_{-\infty}^{\infty} e^{i\lambda t}h(\lambda)d\lambda.$$

Then, as $C(t) = \delta(t)$ we have

$$C(0) = \mathbb{E}[\xi^2(t)] = \int_{-\infty}^{\infty} h(\lambda)d\lambda = \infty,$$

this result contradicts the fact that the variance $C(0) = \mathbb{E}[\xi^2(t)] = 1$ (Definition 2.1(a)). Thus, the white noise Gaussian $\xi(t)$ *not exists (in the usual sense). Therefore, what sense have the equation (3)? The response the have the biologist Robert Brown and the mathematicians Louis Bachelier and Norbert Wiener.*

In 1827 Robert Brown (Scottish biologist) observe a suspended pollen grain in water. While looking at this pollen grain underneath a microscope, he notices that it undergoes a type of random walk. The phenomenon that he observed now is known as Brownian motion. In 1900, Louis Bachelier, French mathematician modeled the stochastic process now called Brownian motion (Definition 2.3), which was part of his PhD thesis The *Theory of Speculation*. In 1923, Norbert Wiener (American mathematician and philosopher) gave the first constructive demonstration of Brownian motion. By this reason the Brownian motion is also known as Wiener process.

Definition 2.3 *(Brownian motion). Any continuous time stochastic process $W(\cdot) := \{W(t), t \leq 0\}$ is called a Wiener process or Brownian motion if satisfies that*

(a) $W(0) = 0$,

(b) *has independent increments, i.e., for all times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ the random variables $W(t_n) - W(t_{n-1}), W(t_{n-1}) - W(t_{n-2}), \ldots, W(t_2) - W(t_1)$, are independent,*

(c) *has stationary increments, that is, the distribution of the increment $W(t+h) - W(t)$ does not depend on t, moreover, $W(t+h) - W(t)$ has a normal distribution with mean zero and variance h,*

(d) *has almost surely continuous paths,*

(e) *is almost surely nowhere differentiable.*

In 1908, Paul Langevin (French physicist) found that the velocity of a particle that move with Brownian motion satisfies the SDE

$$dx(t) = -\alpha x(t)dt + \sigma\xi(t)dt, \ \ \alpha > 0 \ \text{and} \ \sigma = \text{constant}, \tag{6}$$

also, it found that if the Brownian motion $W(t)$ were differentiable, then its derivative would the Gaussian white noise process,

$$dW(t) = \xi(t)dt. \tag{7}$$

So, the Langevin equation (6) can be written as

$$dx(t) = -\alpha x(t)dt + \sigma dW(t). \tag{8}$$

Generally, the SDE $dx(t) = b(t, x(t))dt + \sigma\xi(t)dt, x(0) = x_0$, can be expressed as

$$\begin{aligned} &dx(t) = b(t, x(t))dt + \sigma(t, x(t))dW(t), \\ &x(0) = x_0, \ \ 0 \leq t \leq T, \end{aligned} \tag{9}$$

which in integral form is given by

$$x(t) = x(0) + \int_0^T b(t, x(t))dt + \int_0^T \sigma(t, x(t))dW(t), \tag{10}$$
$$x(0) = x_0, \ \ 0 \leq t \leq T.$$

The first integral in right-hand side is interpreted in the usual sense, but how is interpreted the second integral? In 1951, the Japanese mathematician Kiyosi Itô developed the Itô's calculus whose central concept is the *stochastic integral*. In this work, let us consider an n-dimensional stochastic differential equation (also called diffusion process) evolving according to (9), where $b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n_1}$ are given functions called the *drift* and the *diffusion term*, and $W(\cdot)$ is an n_1-dimensional standard Brownian motion. The following assumption ensures the existence and uniqueness of solutions to the stochastic differential equation (9), see for instance, Mao and Yuen (2006).

Assumption 2.4 *(a)* (*Lipschitz condition.*) *The functions $b(t,x)$ and $\sigma(t,x)$ are continuous on $\mathbb{R}^n$, and $x \mapsto b(t,x)$, $x \mapsto \sigma(t,x)$ satisfies a* Lipschitz condition *uniformly in $t \in [0,T]$; that is, there exist two positive constants K_1 and K_2 such that for all $x, y \in \mathbb{R}^n$*

$$|b(t,x) - b(t,y)| \le K_1|x-y|,$$

$$|\sigma(t,x) - \sigma(t,y)| \le K_2|x-y|.$$

(b) (Linear growth condition.) *There exists a constant K_3 such that for all $x \in \mathbb{R}^n$.*

$$|b(t,x)| + |\sigma(t,x)| \le K_3(1+|x|).$$

Remark 2.5 *(a) If Assumption 2.4 hold, then there exists a solution to stochastic differential equation (9) given by*

$$x(t) = x(0) + \int_0^T b(s,x(s))ds + \int_0^T \sigma(s,x(s))dW(s), \tag{11}$$

where the first integral of the right-hand side is the Lebesgue integral and second is the Itô's stochastic integral.

(b) The equation (11) is known as the Itô's integral equation.

(c) The Itô's stochastic integral does not obey the rules of ordinary calculus, instead of ordinary (classical) calculus we have the Itô's calculus.

Itô's calculus. Let $C^{1,2}([0,T] \times \mathbb{R}^n)$ be the space of real-valued functions $f(t,x)$ on $[0,T] \times \mathbb{R}^n$ which is twice continuously differentiable in x and once differentiable in t. Let $f(t,x)$ be in $C^{1,2}([0,T] \times \mathbb{R}^n)$, we denote by f_x and f_{xx} the gradient (row) vector and the Hessian matrix of f, respectively.

The following lemma shows the well known Itô formula. For a proof we quote Friedman (2007) Theorem 5.3, or Morimoto (2010), Theorem 1.6.2.

Lemma 2.6 (Itô's lemma). *Let $x(\cdot)$ be as in (11) and let $f(t,x) \in C^{1,2}([0,T] \times \mathbb{R}^n)$. Then, the stochastic process $y(t) = f(t,x(t))$ satisfies the stochastic differential equation*

$$dy(t) = f_t(t,x(t))dt + f_x(t,x(t))dx(t) + \frac{1}{2} f_{xx}(t,x(t))(dx(t))^2. \tag{12}$$

Substituing (9) in (12) and using the rule of multiplication $dt \cdot dt = 0$, $dt \cdot dW = 0$, $dW \cdot dt = 0$ and $dW \cdot dW = dt$, we obtain

$$\begin{aligned} dy(t) =& [f_t(t,x(t)) + b(t,x(t)) f_x(t,x(t)) + \frac{1}{2} f_{xx}(t,x(t))]dt \\ &+ f_x(t,x(t))\sigma(t,x(t))dW(t). \end{aligned} \tag{13}$$

The stochastic differential equation (13) is called the Itô's formula and it is an extension to the stochastic theory of the rule chain of ordinary calculus.

3 Applications

In this section it shows some applications of the stochastic differential equations in the modeling from an electrical RLC circuit, a liquid level system and a DC motor. The evolution of these systems are modeled by means a SDE, as well as, an ODE with the aim to compare both solutions and so, can evaluate if the solution (trajectory) of an SDE presents a better approximation to the experimentally measured trajectories than the solution of an ODE.

3.1 Series RLC circuit

This section presents the ordinary differential equation and the stochastic differential equation, which model the charge in a series RLC circuit. In the rest of this work we consider a source of voltage of direct current (DC).

A series RLC circuit, as shown in the Fig.1, is a single electric network consisting of a Resistance (R), an Inductance (L), a Capacitance (C) and a source of voltage (V).

Hayt et al. (2012) or Nilsson and Reidel (2015). The *Kirchhoff's voltage law*, gives the ordinary differential equation that describes the behavior of the current $i(t)$ in the circuit:

$$V = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau, \quad t \ge 0. \tag{14}$$

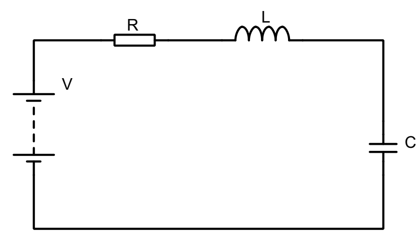

Fig. 1: Series RLC circuit.

Deterministic Model. Let $q(t)$ be the charge in the circuit (see Fig.1) at time t. We know that $i(t) = \frac{dq(t)}{dt}$, then we can rewrite (14) as

$$V = R\frac{dq(t)}{dt} + L\frac{d^2q(t)}{dt^2} + \frac{1}{C}q(t) \tag{15}$$

with initial conditions $q(0) = 0$ and $q'(0) = 0$. Multiplying by $1/L$ the equality (15) and rearranging terms, we have

$$\frac{d^2q(t)}{dt^2} = -\frac{1}{LC}q(t) - \frac{R}{L}\frac{dq(t)}{dt} + \frac{V}{L}, \tag{16}$$

which indicates the evolution of the charge in the series RLC circuit.

If we define the column vector $\boldsymbol{Q}(t)$ as

$$\boldsymbol{Q}(t) := \begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix} = \begin{pmatrix} q(t) \\ q'(t) \end{pmatrix} \tag{17}$$

then, from (16) the charge in the circuit will be represented by the matrix differential equation

$$\frac{d\boldsymbol{Q}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{V}{L} \end{bmatrix}. \tag{18}$$

Taking $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ \frac{V}{L} \end{bmatrix}$, of (16)-(18) we obtain the lineal ordinary differential equation for the charge in a series RLC circuit

$$\frac{d\boldsymbol{Q}(t)}{dt} = \boldsymbol{A}\boldsymbol{Q}(t) + \boldsymbol{b}, \tag{19}$$

with analytic solution

$$\boldsymbol{Q}(t) = e^{\boldsymbol{A}t}\boldsymbol{Q}_0 + \left[\int_0^t e^{\boldsymbol{A}(t-z)}\,dz\right]\boldsymbol{b} \quad \text{where} \quad \boldsymbol{Q}_0 = \begin{pmatrix} q(0) \\ q'(0) \end{pmatrix}. \tag{20}$$

Stochastic model. The most used DC sources currently are switched sources. This last sources is more efficient than regulated sources, but generate high frequency noise. Since the system is affected by the voltage sources noise, it can be modeled through stochastic differential equations. Considering a stochastic effect in the source, the voltage can be represented by

$$V^*(t) = V(t) + \alpha\xi(t), \tag{21}$$

where $\xi(t)$ is the white noise and α a constant (that in our case is determined of the experimental data). Substituting (21) in the equation (15) and proceeding as in the deterministic model we can get

$$\frac{d\boldsymbol{Q}(t)}{dt} = \boldsymbol{A}\boldsymbol{Q}(t) + \boldsymbol{b} + \xi(t)\boldsymbol{a}, \tag{22}$$

with $\boldsymbol{a} = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix}$ and $\boldsymbol{A}$, $\boldsymbol{b}$ as in (19).

Now, using (7), the matrix stochastic differential equation (22) in terms of the Brownian motion is equal to

$$d\boldsymbol{Q}(t) = (\boldsymbol{A}\boldsymbol{Q}(t) + \boldsymbol{b})dt + \boldsymbol{a}dW(t). \tag{23}$$

The equation (23) is a lineal stochastic differential equation (SDE) with constant coefficients. It's easy to prove that this lineal SDE satisfies the Assumption 2.4, therefore, there exists a solution $\boldsymbol{Q}(\cdot)$ which is calculated with Itô's formula (13) as follows.

Solution with Itô's Formula. To find the analytic solution of the equation (23) with Itô's formula first, we define $g(t, \boldsymbol{Q}(t)) = e^{-\boldsymbol{A}t}\boldsymbol{Q}(t)$, therefore, Itô's differential rule as in Lemma 2.6 gives

$$dg(t, \boldsymbol{Q}(t)) = -\boldsymbol{A}e^{-\boldsymbol{A}t}\boldsymbol{Q}(t)dt + e^{-\boldsymbol{A}t}d\boldsymbol{Q} + \tfrac{1}{2}(0)(d\boldsymbol{Q})^2, \tag{24}$$

replacing $d\boldsymbol{Q}$

$$\begin{aligned} d(e^{-\boldsymbol{A}t}\boldsymbol{Q}(t)) &= -\boldsymbol{A}e^{-\boldsymbol{A}t}\boldsymbol{Q}(t)dt \\ &\quad + e^{-\boldsymbol{A}t}((\boldsymbol{A}\boldsymbol{Q}(t) + \boldsymbol{b})dt + \boldsymbol{a}dW(t)) \\ &= e^{-\boldsymbol{A}t}\boldsymbol{b}dt + e^{-\boldsymbol{A}t}\boldsymbol{a}dW(t), \end{aligned} \tag{25}$$

integrating the last equation we get

$$\boldsymbol{Q}(t) = e^{\boldsymbol{A}t}\boldsymbol{Q}_0 + \left[\int_0^t e^{\boldsymbol{A}(t-z)}dz\right]\boldsymbol{b} + \left[\int_0^t e^{\boldsymbol{A}(t-z)}dW(z)\right]\boldsymbol{a}. \tag{26}$$

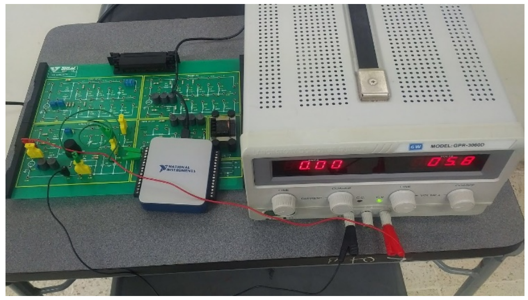

Fig. 2: Series RLC circuit and NI data acquisition card USB 6003.

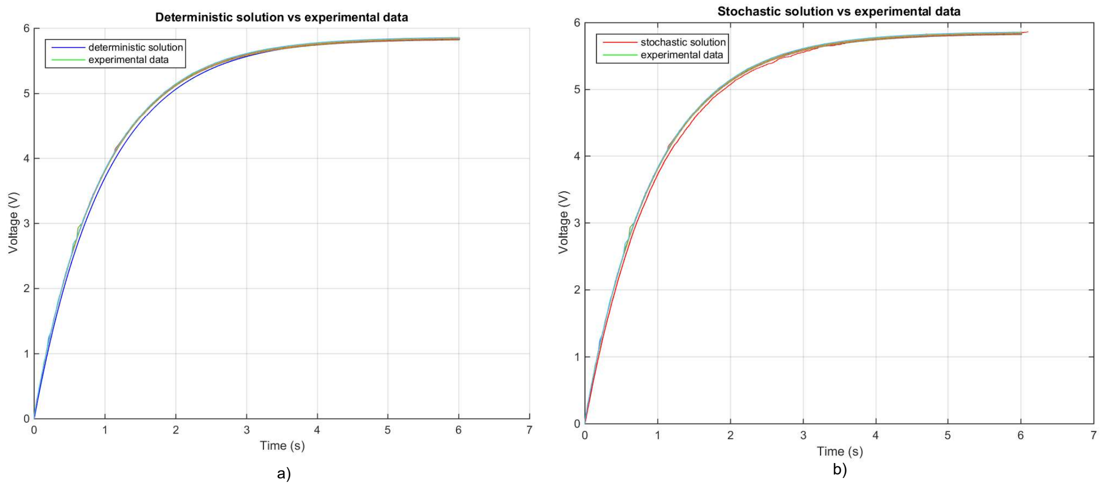

Fig. 3: Voltage experimental data, deterministic and stochastic voltage.

Remark 3.1 *The Euler-Maruyama method us gives a numerical approximation for* $\boldsymbol{Q}(t)$ *in (26), i.e., it gives an approximation for the charge* $q(t)$ *and the current* $\frac{dq(t)}{dt} = i(t)$*. But, since in this section a stochastic effect on the voltage source is considered, in the following two subsections the formula* $V(t) = \frac{dq(t)}{C}$ *is used for modelling the voltage through of the RLC circuit. Moreover, It will be written* deterministic solution *(deterministic voltage) when the approximation* $q(t)$ *is modeled by ODE (20) and will be writing* stochastic solution *(stochastic voltage) when* $q(t)$ *let be approximate by the SDE (26).*

3.1.1 Experimental procedure

A DC voltage source connected to a circuit was used to carry out the experiment of RLC serie electric circuit, with $V = 5.8V$, $R = 100k\Omega$, $L = 336mH$, and $C = 10\mu F$, as shown in Fig.1. It was used a National Instrument (NI) data acquisition card USB 6003 with a resolution of 16 bits connected to Labview software to measure the voltage across the capacitor and store data. The experiment consists of connecting the DC source to the RLC circuit and measuring the voltage in the capacitor, the source is disconnected and the capacitor is discharged, this process was repeated 50 times, see Fig.2.

3.1.2 Numerical results

This section shows the graphs of the voltage experimental data in the capacitor of RLC circuit, as well as, the deterministic and stochastic solutions of the ordinary differential equation and stochastic differential equation, respectively, which model this circuit, see Remark 3.1.

Table 1

Table 1. Mean squared errors for the RLC circuit

	Mean squared errors
Stochastic solution vs experimental data	0.38%
Deterministic solution vs experimental data	0.47%

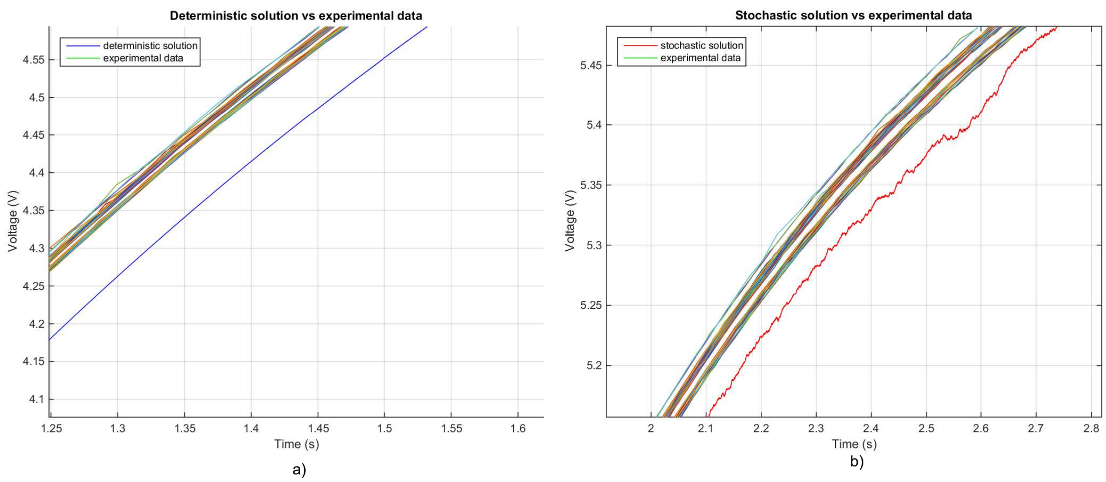

Fig. 4: Amplification of the deterministic and stochastic voltages vs voltage measurements.

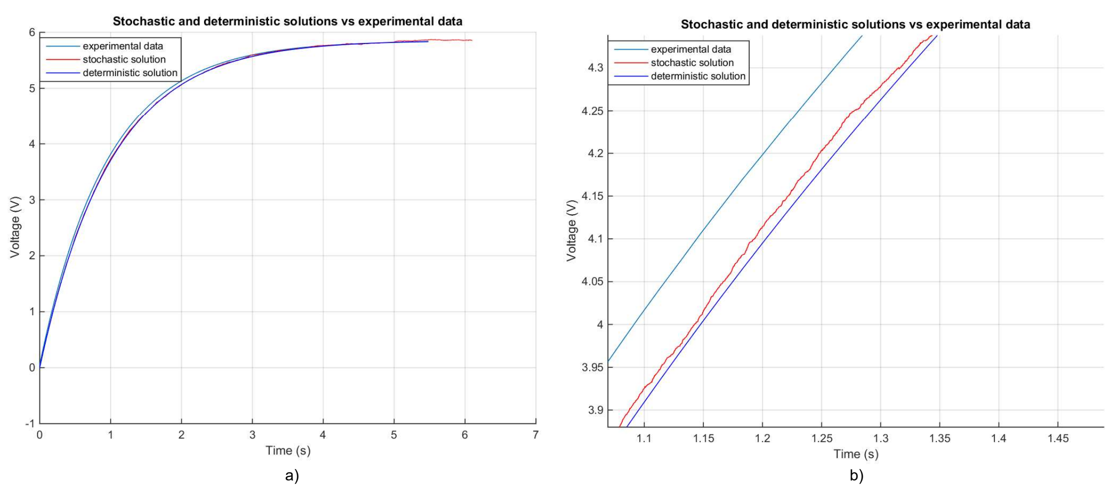

Fig. 5: Comparison between the voltage experimental data and the deterministic and stochastic voltage.

The voltages in the capacitor measured in the 50 experiments that were carried out are compared with the deterministic (analytical) solution of the ODE (20), see Fig. 3 a). There are slight differences, this is due to the variations of noise that exists in the voltage source. Using the Euler's numerical method we simulated the stochastic differential equation (26) with α = 0.0277. The α value was estimated directly from measured data, see Fig. 3 b).

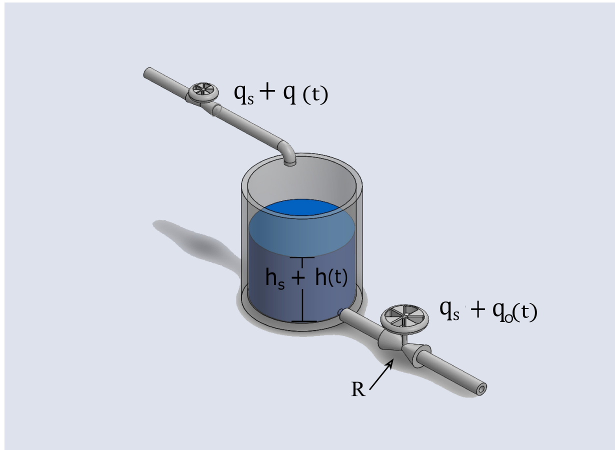

Fig. 6: Liquid level system.

We observed in Fig. 4 a)-b) an amplification that is shown to detail the ODE solution, the SDE solution and the voltage measurements in the capacitor. Fig. 5 show the stochastic and deterministic solutions vs. experimental data. It notes that the SDE solution is closer to the experimental data. In fact, the mean squared error E_{ms} was calculated, see Table 1. In this table, we can see that the stochastic solution is closer to the experimental data because the mean squared error (MSE) presents a minor value, 0.38 % in comparison with the MSE between the experimental data and deterministic solution, 0.47 %.

$$E_{ms} = \frac{1}{n}\sum_{i=1}^{n}(\widehat{Y}_i - Y_i)^2, \tag{27}$$

where $\widehat{Y}_i$ is a vector of n predictions and Y_i is the average vector of the measured values.

3.2 Liquid level system

Industrial processes the most common variables to control are flow and level, for example, in the process of the petrochemical industries and generation of energy, usually required a constant level in their process for example in the case of a boiler that generates pressure steam, usually need a level constant in a tank to prevent it from running out of water and overpressure occurs.

The system shown in Fig. 6 consists of a tank of uniform cross section area A, which has two valves, a control valve in the inflow and a load valve in the outflow. The load valve represents a resistance (R) for the outflow. In this application we are interested in to analyze the variation over time of the height of tank level, $h(t)$, from its steady-state value, h_s, when steady-state flow rate, q_s, present a small deviation, $q(t)$. For this end, we define the following variables.

- q_s= steady-state flow rate (before any change has occurred), m^3/s
- $q(t)$ = small deviation of inflow rate from its steady-state value, m^3/s
- $q_o(t)$ =small deviation of outflow rate from its steady-state value (flow rate (volume/time) through the resistance), m^3/s
- h_s =steady-state head (before any change has occurred), m
- $h(t)$ =small deviation of head from its steady-state value, m

As mentioned the authors in Ogata (2010), the relationship between R, $q_o(t)$ and $h(t)$ is given by

$$R = \frac{h(t)}{q_o(t)}, \tag{28}$$

for a laminar flow. Then, using a transient mass balance around the tank:

[rate of mass flow in] – [rate of mass flow out]
= [Rate of accumulation of mass in tank],

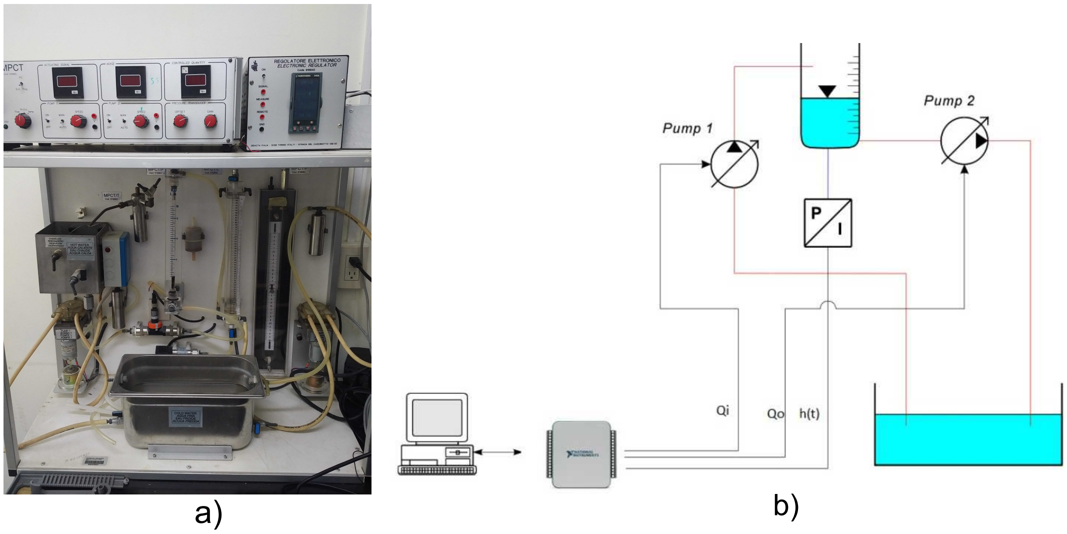

Fig. 7: Modular Process Control Trainer (MPCT ®) and level process diagram.

it obtains

$$\rho q(t) - \rho q_o(t) = \rho A \frac{dh(t)}{dt} \quad (29)$$

$$q(t) - q_o(t) = A \frac{dh(t)}{dt}, \quad (30)$$

where ρ is the liquid density. Thus, from (28) and (30) it have for a constant value of R, the following ordinary linear differential equation:

$$dh(t) = \Big(\frac{q(t)}{A} - \frac{h(t)}{RA}\Big)dt, \quad (31)$$

which models the variation of $h(t)$ in a small time interval, dt.

Stochastic differential equation. Now consider that small deviation of inflow rate $q(t)$ is the form

$$q^*(t) = q(t) + \beta\xi(t). \quad (32)$$

So, replacing (32) in (30) a simple calculus gives

$$dh(t) = \Big(\frac{q(t)}{A} - \frac{h(t)}{RA}\Big)dt + \frac{\beta}{A}dW(t). \quad (33)$$

3.2.1 Experimental procedure

Measurements were taken from the Modular Process Control Trainer (MPCT) of the Universidad Veracruzana campus Coatzacoalcos is shown in the Fig. 7. The level process is described in Fig. 7(b) as follows: the signal $Q_i := q_s + q(t)$ is transmitted to peristaltic pump 1 which adjusts the inflow to the graduated tank accordingly. The signal $Q_o := q_s + q_0(t)$ is transmitted to peristaltic pump 2 which adjusts the outflow from the graduated tank. The signal corresponding to the tank level, $h(t)$, is supplied by the pressure transducer fitted to the bottom of the tank. The pressure and level are correlated according to a linear relationship. These signals are connected to data acquisition card USB -6003, through a USB protocol connected to the PC with software Labview where the data is stored.

A total of 50 measurements was performed on the MCPT to measure the level of the graduated tank with initial conditions $h_s(t) = 3.33$ %, $Q_i = 6.1$ l/h and $Q_o = 5.1$ l/h. The graphs of the mean of experimental measures versus the solution deterministic are shown in the Fig.8 a)- b), where the differences that exist in the measurements of the level with respect to the time are generated for the inflow Q_i caused by the inherent characteristic of the operation of the peristaltic pumps.

Remark 3.2 *The variations (or noise) in the flow is due to two-shoe peristaltic pump belongs to positive displacement pump class which pump fluid at a constant average flow rate. However, because the individual pumping elements of these pumps discharge discrete quantities of fluid, the instantaneous flow rate varies in cyclic form. Pulsations are observed in the system as pressure spikes. This pulsating flow can cause operational problems and shorten equipment service life.*

Table 2. Mean squared errors for the level system

	Mean squared errors
Stochastic solution vs experimental data	4.57%
Deterministic solution vs experimental data	6.32%

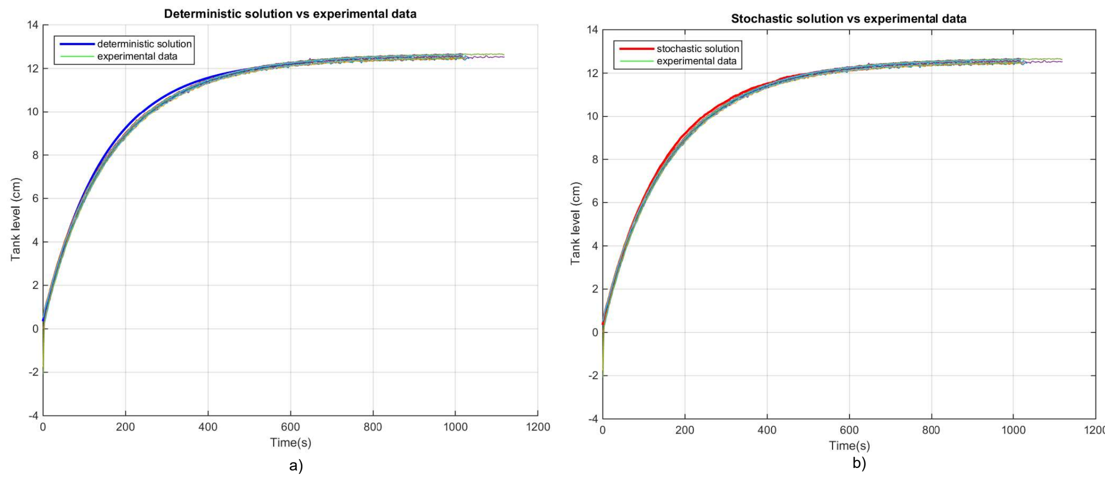

Fig. 8: Level experimental data, stochastic and deterministic levels.

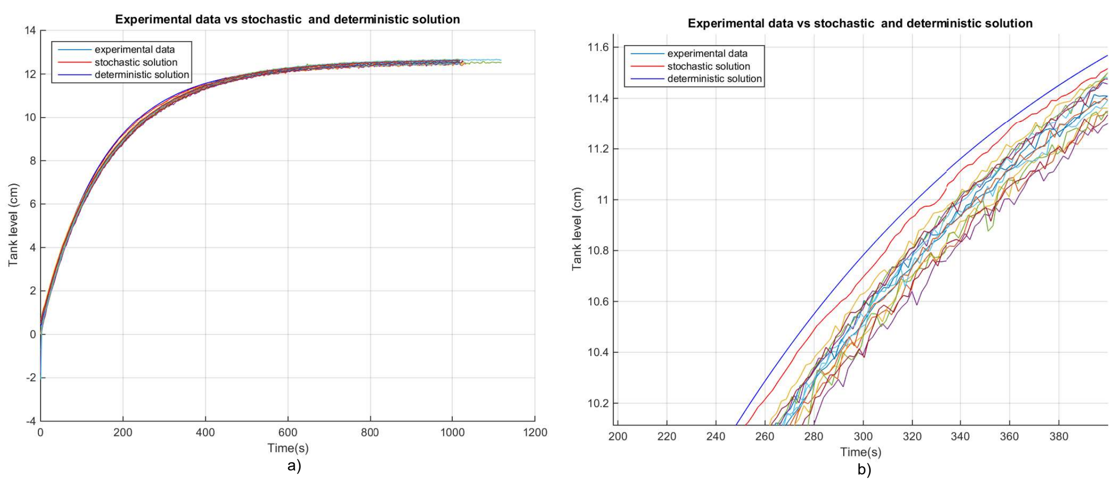

Fig. 9: Amplification of level experimental data vs stochastic and deterministic levels.

3.3 Numerical results

Experiments were performed to measure the level of the liquid in the tank, the deterministic (31) and stochastic solutions (33) are shown in the Fig. 8 a)- b), respectively. Fig. 9 a) we plot the level experimental data and deterministic and stochastic solutions (levels), whereas, Fig. 9 b) an amplification of the simulations is done. A visual inspection allows us to ensure that the solution of the SDE is closer to the level experimental data that the solution of the ODE. In fact, calculating the mean squared errors E_{ms}, it possible to confirm this observation, see Table 2.

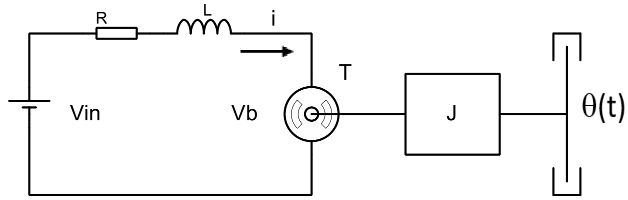

Fig. 10: DC motor.

3.4 Angular velocity in a DC motor

This application, it analyses the differential equation that model the angular velocity in a DC motor. Our main source is Emhemed and Mamat (2012).

The DC motors are devices that convert electrical energy into mechanical energy, they are formed by two parts the stator and the rotor (induced). Mathematical modeling of the DC motor requires two equations, a mechanical equation and other electrical equation. These equations are coupled and are based on the laws of dynamics and Kirchhoff, respectively. The mechanical equation, mainly models the movement of the rotor, and the electric equation models what happens in the electrical circuit of the armature, Mora (2008).

The motor model has electrical variables that are: supply voltage of the rotor, Vin, current i that will circulate by the rotor (armature current), winding resistance of the rotor R, and the inductance of the winding of the rotor L; the mechanical characteristics are: angular velocity of rotation of the rotor ω, moment of inertia equivalent of the rotor shaft J and the angular position θ as shown in Fig. 10.

Let K_T and K_b be proportionality constants. Is well known that the torque T, available at the shaft of a DC motor is proportional to the current i, i.e.,

$$T = K_T i \tag{34}$$

and the voltage $V_b(t)$, across the rotor is proportional to the angular velocity of the shaft, that is,

$$V_b = K_b \frac{d\theta}{dt} \tag{35}$$

Using the Kirchhoff's voltage law for the motor circuit given in Fig. 10, we obtain

$$V_{in} = Ri + L\frac{di}{dt} + V_b, \tag{36}$$

replacing (40) and (39) in (36)

$$V_{in} = \frac{R}{K_T}T + \frac{L}{K_T}\frac{dT}{dt} + K_b\frac{d\theta}{dt}. \tag{37}$$

The relation between torque and angular acceleration for a shaft whose moment of the inertia is J,

$$T = J\frac{d^2\theta}{dt} + V_b, \tag{38}$$

us allows obtaining the third-order differential equation that model the angular acceleration in a DC motor.

$$V_{in} = \frac{JL}{K_T}\frac{d^3\theta}{dt^3} + \frac{RJ}{K_T}\frac{d^2\theta}{dt^2} + K_b\frac{d\theta}{dt}. \tag{39}$$

For many motors, the inductance can be neglected. So, the equation (39) can be rewritten as

$$V_{in} = \frac{RJ}{K_T}\frac{d^2\theta}{dt^2} + K_b\frac{d\theta}{dt} \;\Rightarrow\; \frac{d^2\theta}{dt^2} = -\frac{K_T K_b}{RJ}\frac{d\theta}{dt} + \frac{K_T}{RJ}V_{in} \tag{40}$$

Taking $\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{K_T K_b}{RJ} \end{bmatrix}$, $\boldsymbol{b}_1 = \begin{bmatrix} 0 \\ \frac{K_T}{RJ}V_{in} \end{bmatrix}$ and $\Theta(t) := \begin{pmatrix} \Theta_1(t) \\ \Theta_2(t) \end{pmatrix} = \begin{pmatrix} \theta(t) \\ \frac{d\theta(t)}{dt} \end{pmatrix}$. The matrix representation of the differential equation (40) is

$$\frac{d\Theta(t)}{dt} = \boldsymbol{A}_1\Theta + \boldsymbol{b}_1, \tag{41}$$

which is a linear differential equation similar to the given for the RLC circuit, see (19). Again, considering noise in the voltage source, $V_{in}(t) = V_{in}(t) + \alpha\xi$, it can get the stochastic differential equation which model the angular acceleration in a DC motor as in Section *3.1*, obtaining

$$d\Theta(t) = (\boldsymbol{A}_1\Theta + \boldsymbol{b}_1)dt + \boldsymbol{a}_1 dW(t) \tag{42}$$

with $\boldsymbol{a}_1 = \begin{pmatrix} 0 \\ \frac{\alpha K_T}{RJ} \end{pmatrix}$. The Itô's solution for the SDE (42) is (26) taking $\boldsymbol{Q} \equiv \Theta$, $\boldsymbol{A} \equiv \boldsymbol{A}_1$, $\boldsymbol{b} \equiv \boldsymbol{b}_1$ and $\boldsymbol{a} \equiv \boldsymbol{a}_1$, i.e.,

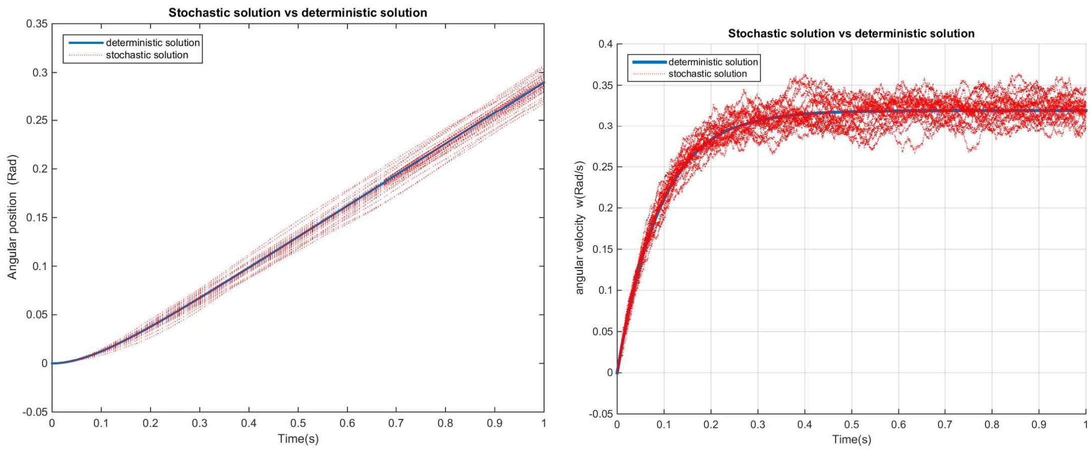

Fig. 11: Angular position and velocity, θ and $\frac{d\theta}{dt}$, respectively.

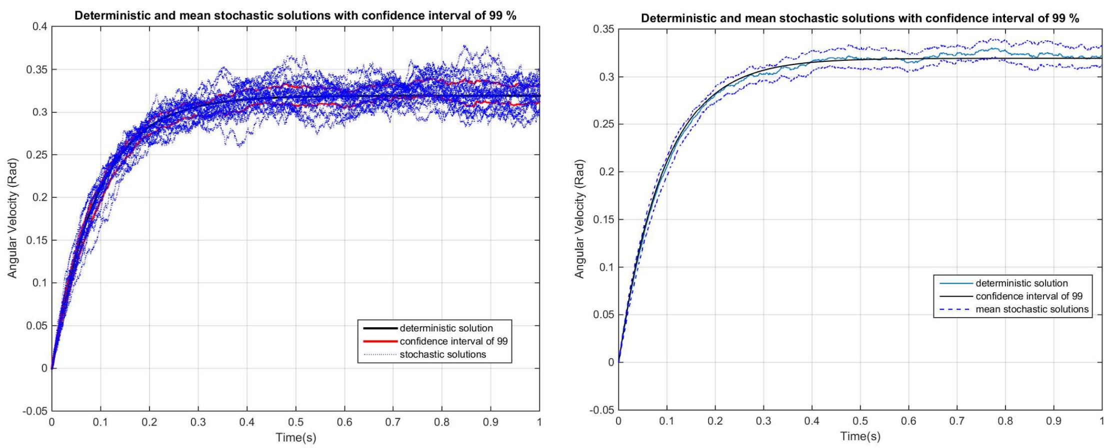

Fig. 12: Angular velocity.

$$\Theta(t) = e^{\boldsymbol{A}_1 t}\Theta_0 + \left[\int_0^t e^{\boldsymbol{A}_1(t-z)}dz\right]\boldsymbol{b}_1 + \left[\int_0^t e^{\boldsymbol{A}_1(t-z)}dW(z)\right]\boldsymbol{a}_1. \quad (43)$$

Note that, taking expected value in both side of (43) it get

$$\begin{aligned}\mathbb{E}[\Theta(t)] &= \mathbb{E}[e^{\boldsymbol{A}_1 t}\Theta_0] + \mathbb{E}\Big[\left[\int_0^t e^{\boldsymbol{A}_1(t-z)}dz\right]\boldsymbol{b}_1\Big] \\ &\quad + \mathbb{E}\Big[\left[\int_0^t e^{\boldsymbol{A}_1(t-z)}dW(z)\right]\boldsymbol{a}_1\Big] \quad (44) \\ &= e^{\boldsymbol{A}_1 t}\Theta_0 + \left[\int_0^t e^{\boldsymbol{A}_1(t-z)}dz\right]\boldsymbol{b}_1. \quad (45)\end{aligned}$$

The equality (44) is because the expectation of the Itô integral is zero and the expectation of a constant is the same constant. So, (44) implies that the expected value of the stochastic solution coincides with the deterministic solution (20) independently of the noise amplitude. In the following section this statement is verified from the sample mean of the stochastic solutions as an estimator of the expectation, $\mathbb{E}[\Theta(t)]$. Confidence intervals for this estimator are calculated using the Student's t-distribution since we can suppose that the stochastic solution it's normally distributed.

Remark 3.3 *Noise in voltage sources is due to factors such as ripple voltage which is the alternating component of the unidirectional voltage from a rectifier or generator used as a source of DC.*

This ripple is due to incomplete suppression of the alternating waveform after rectification. Ripple voltage originates as the output of a rectifier or from the generation and commutation of DC power.

3.5 Numerical results

The simulation of the DC motor was carried out using the following parameters $R = 5.2\Omega$, $J = 0.01 kg.m$, $K_T = 0.75 N.mA^{-1}$, $K_b = 0.75 N.mA^1$ and $Vin = 24V$. Data were taken from Emhemed and Mamat (2012).

In Fig. 11 we can see the solutions (angular position and velocity) of the ODE (41) and SDEs (42) for different amplitudes of the Brownian motion. Fig. 12 its plot the sample mean of the stochastic solutions and deterministic solution with an interval confidence of 99%, as can see, the sample mean of the stochastic solution is close the deterministic solution as indicated the theory, (44).

4 Stochastic optimal control

This section describes the stochastic optimal control theory to illustrate of the use from SDEs. To this end, It's study an optimal investment problem in which the stochastic calculus and the optimal control theory are applied, Cklebaner (2005), Durret (1996), Fleming and Rishel (1975), Ghosh et al. (1992), Ghosh et al. (1997) . In the optimal investment problem a stochastic differential equation with Markovian switching which modelling the price of a stock in a financial market is considered. The form in which the optimal control problem is resolved can be applied to any optimal control problem in engineering, of course, with its respective modifications.

In an optimal control problem must be indicate the:

(i) evolution of the dynamic system (48),

(ii) controls (actions) set available for the controller (51),

(iii) objective function (function to optimize) (53), and

(iv) some additional restrictions (50)-(51),

with these four elements, the optimal control problem consists with to optimize (minimize/maximize) the objective function over the controls set.

We use the dynamic programming technique applied to continuous-time optimal stochastic problem which is based on *Bellman's principle of optimality*. The value function v (54), is defined as the minimun cost /maximun payoff (minimun/maximum objective function). Assuming that v is a continuously differentiable function, then application of the principle of optimality yields the so-called *Hamilton-Jacobi-Bellman (HJB) equation* (55). If v can be found, then the HJB equation (55) is provided by means of obtaining the optimal control.

The control system we are concerned with is the controlled Markov-modulated diffusion process (also known as a piecewise diffusion or a switching diffusion or a diffusion with Markovian switching)

$$dx(t) = b(x(t), \psi(t), u(t))dt + \sigma(x(t), \psi(t))dW(t),$$
$$x(0) = x,\ \psi(0) = i, \tag{46}$$

with coefficients depending on a continuous-time irreducible Markov chain $\psi(\cdot)$ with a finite state space $E = \{1, 2, \ldots, N\}$, and transition probabilities

$$\mathcal{P}(\psi(s+t) = j | \psi(s) = i) = q_{ij}t + o(t). \tag{47}$$

For states $i \neq j$ the number $q_{ij} \geq 0$ is the transition rate from i to j, while $q_{ii} = -\sum_{j\neq i} q_{ij}$. Moreover, in (46), $b : \mathbb{R}^n \times E \times U \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times E \to \mathbb{R}^{n\times d}$ are given functions, usually called the drift and the dispersion matrix, respectively, and $W(\cdot)$ is a d-dimensional standard Brownian motion independent of $\psi(\cdot)$. The stochastic process $u(\cdot)$ is a $U-$valued process called the control process and the set $U \subset \mathbb{R}^m$ is called the control (or action) space.

4.1 Optimal investment model

In mathematical finance, the price of a stock is often modeled as a geometric Brownian motion which is determined by two parameters: μ (expected return rate) and σ (volatility). (A detailed analysis of this model appears, for instance, in Karatzas and Shreve (1998).) These parameters are usually considered to be deterministic. As Bäuerle and Rieder (2004) point out, models with deterministic coefficients are only good for relatively short periods of time and cannot respond to changing conditions. Some factors that influence the movement of stock prices are the market changing conditions due to external factors, such as inflation, money devaluation, and so forth. To incorporate the trends of the stock (up or down) due to external factors, it is necessary to modify the geometric Brownian motion to allow for the expected return rate and the volatility to depend on general market conditions (for further motivation see, for instance, Zhang (2001)). This is the financial market considered in this application, namely, a Black-Scholes market with Markovian switchings (also called a Markov-modulated market), so that the coefficients depend on a continuous-time finite-state homogeneous Markov chain. The states of the Markov chain represent the market conditions.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and $T > 0$ a fixed time horizon. Let $W_1(\cdot)$ and $W_2(\cdot)$ be standard Brownian motions. The price process $P(\cdot)$ of the risky stock satisfies the stochastic differential equation with Markovian switchings

$$dP(t) = P(t)\mu(\psi(t))dt + P(t)\sigma(\psi(t))dW_1(t) \text{ for } t \in [0,T], \tag{48}$$

while the stochastic cash flow, or risk process of the company, satisfies that

$$dY(t) = \alpha dt + \beta dW_2(t) \ \ \forall\, t \in [0,T], \tag{49}$$

where $\alpha \in \mathbb{R}$ and $\beta > 0$ are given constants. It assume that the functions $\mu(\cdot), \sigma(\cdot) : E \to \mathbb{R}$ in (48) satisfy that

$$\int_0^T \left\{ |\mu(\psi(t))| + |\sigma(\psi(t))| \right\} dt < \infty \ \text{ a.s.}, \tag{50}$$

and, we assume that $\psi(\cdot)$ and $W_j(\cdot)$ are independent ($j = 1, 2$), but the Brownian motions $W_1(\cdot)$ and $W_2(\cdot)$ can be correlated, with a correlation coefficient $|\rho| < 1$, that is, $\mathbb{E}[W_1(t)W_2(t)] = \rho t$. We ignore the uninteresting case $\rho^2 = 1$, because then there would be only one source of randomness in the model.

We denote by $f(t)$ the total amount of money invested by the company in the risky stock at time t under an investment strategy f. The set of admissible strategies f over the planning period $[t,T]$, for every $0 \le t \le T$, is given by

$$\mathcal{A}(t, \widehat{\mathcal{F}}_t) := \{ f : [t,T] \to \mathbb{R} \mid f \text{ is } \widehat{\mathcal{F}}_t - \text{adapted}, \int_t^T f(s)^2 ds < \infty \ \text{ a.s.} \}, \tag{51}$$

where $\widehat{\mathcal{F}}_t$ is the filtration generated by W_1, W_2 and ψ, i.e., $\widehat{\mathcal{F}}_t = \sigma(W_1(s), W_2(s), \psi(s),\ s \le t)$ for $t \ge 0$.

Let $X^f(t)$ be the wealth of the company at time t, given that it follows the strategy f, with $X^f(0) = x_0$ being the initial wealth. This process then, by (48) and (49), evolves as

$$\begin{aligned} dX^f(t) &= f(t)\frac{dP(t)}{P(t)} + dY(t) \\ &= (f(t)\mu(\psi(t)) + \alpha)dt + f(t)\sigma(\psi(t))dW_1(t) \\ &\quad + \beta dW_2(t). \end{aligned} \tag{52}$$

This is a stochastic differential equation with Markovian switchings, which satisfies the Itô's conditions, Assumption 2.4, Mao and Yuan (2006); hence, it has a unique strong solution $X^f(t)$.

Once the company has decided to invest, it has the following problem.

Optimal investment problem. Find a strategy f^* that maximize the utility of the terminal wealth, i.e., find f^* such that

$$\sup_{f \in \mathcal{A}(t,\widehat{\mathcal{F}})} \mathbb{E}^f_{t,x,i}[u(X^f(T))] = \mathbb{E}^f_{t,x,i}[u(X^{f^*}(T))] < \infty \quad \forall\, (t,x,i) \in [0,T] \times \mathbb{R}^+ \times E, \tag{53}$$

where $u : \mathbb{R}^+ \to \mathbb{R}$ is a given utility function and $\mathbb{E}^f_{t,x,i}$ denotes the conditional expectation given $X^f(t) = x$ and $\psi(t) = i$.

Denote by

$$v(t,x,i) := \sup_{f \in \mathcal{A}(t,\widehat{\mathcal{F}})} \mathbb{E}^f_{t,x,i}[u(X^f(T))]. \tag{54}$$

To solve the optimal investment problem we will use the dynamic programming technique. The corresponding HJB equation (Fleming and Rishel (1975) or Karatzas and Shreve (1998)) is

$$\begin{cases} \sup_{y \in M} \mathcal{L}^y v(t,x,i) = 0, & t < T; \\ v(T,x,i) = u(x) \end{cases} \tag{55}$$

with M a compact set, v in $C^{1,2}([0,T] \times \mathbb{R}^+ \times E)$, and $\mathcal{L}^y v(t,x,i)$ as

$$\begin{aligned} \mathcal{L}^y v(t,x,i) &:= v_t(t,x,i) + [y\mu(i) + \alpha]v_x(t,x,i) \\ &+ \frac{1}{2}[y^2\sigma^2(i) + \beta^2 + 2y\sigma(i)\beta\rho]v_{xx}(t,x,i) \\ &+ \sum_{j=1}^N q_{ij} v(t,x,j). \end{aligned} \tag{56}$$

In our model, a solution of the HJB-equation (55) gives us the optimal value function $v(t,x,i)$ and the optimal portfolio strategy f^*. This is a consequence of the following "verification theorem", whose proof is quite standard (see, for instance, Bäuerle and Rieder (2004)).

Theorem 4.1 (*A verification theorem.*) *Suppose that* $G \in C^{1,2}([0,T] \times \mathbb{R}^+ \times E)$ *is a solution of the HJB-equation* (55), *and satisfies the growth condition* $|G(t,x,i)| < K(1 + |x|^k)$ *for some constants* $K > 0$ *and* $k \ge 1$, *and for all* $i \in E$ *and* $0 \le t \le T$. *Then*

(*a*) $G(t,x,i) \ge v(t,x,i)$ *for all* $(t,x,i) \in [0,T] \times \mathbb{R}^+ \times E$.

(*b*) *Let* $y^*(t,x,i)$ *be a maximizer of (56), and let* $f^*(t) := y^*(t,X(t),\psi(t))$*. Then* $G(t,x,i) = v(t,x,i)$ *for all* $x \in \mathbb{R}^+$ *and* $i \in E$*. Moreover,* f^* *is an optimal portfolio strategy.*

Maximizing exponential utility of terminal wealth. We look a strategy that maximizes the expected utility of the terminal wealth of the company when the utility function is of the form

$$u(x) = \lambda - \frac{\gamma}{\theta}e^{-\theta x}, \tag{57}$$

where γ, λ, and θ are all positive constants. The constant θ is called the absolute risk aversion parameter (see, for instance Gerber (1979)).

From Theorem 4.1 we obtain the following result which gives in explicit form the optimal investment strategy that solves optimal investment problem when the utility function $u(\cdot)$ is as in (57).

Theorem 4.2 *The strategy that maximizes the expected utility* (53) *is to invest, at each time* $0 \le t \le T$*, the amount*

$$f^*(t,x,i) = \frac{\mu(i)}{\sigma^2(i)\theta} - \frac{\rho\beta}{\sigma(i)}. \tag{58}$$

The corresponding value function is given by

$$v(t,x,i) = \lambda - \frac{\gamma}{\theta}g(t,i)\exp\{-\theta x\}, \tag{59}$$

where $g(t,i)$ *satisfies the linear differntial equation*

$$g_t(t,i) + g(t,i)a(i) + \sum_{j=1}^{N} q_{ij}g(t,j) = 0, \tag{60}$$

with terminal condition $g(T,i) = 1$*, and* $a(i)$ *is defined as*

$$a(i) := \frac{1}{2}\beta^2(1-\rho^2)\theta^2 - [\alpha - \frac{\rho\beta\mu(i)}{\sigma(i)}]\theta - \frac{1}{2}\Big(\frac{\mu(i)}{\sigma(i)}\Big)^2 \quad \forall\, i \in E. \tag{61}$$

Proof. We suppose that a solution $G \in C^{1,2}([0,T] \times \mathbb{R}^+ \times E)$ of the HJB-equation (55) has the form

$$G(t,x,i) = \lambda - \frac{\gamma}{\theta}g(t,i)e^{-\theta x}, \tag{62}$$

where la function $g(\cdot,i)$ is in $C^1([0,T])$ for all $i \in E$. We also assume that $g \ge 0$, so that the function $x \mapsto G(t,x,i)$ is concave. Therefore we have

$$G_t = -\frac{\gamma}{\theta}e^{-\theta x}g_t,\; G_x = \gamma g e^{-\theta x},\; G_{xx} = -\gamma\theta g e^{-\theta x}. \tag{63}$$

Inserting these values in (55) we see that the maximizer f^* of the HJB-equation is given by the function

$$f^*(t,x,i) = -\frac{\mu(i)}{\sigma^2(i)}\Big(\frac{G_x(t,x,i)}{G_{xx}(t,x,i)}\Big) - \frac{\rho\beta}{\sigma(i)}. \tag{64}$$

By (63) and (64) it follows that f^* is an admissible portfolio strategy. Moreover, substituting (63) and (64) in (55) gives

$$g_t(t,i) + g(t,i)a(i) + \sum_{j=1}^{N} q_{ij}g(t,j) = 0, \tag{65}$$

with terminal condition $g(T,i) = 1$ for $i \in E$, and $a(i)$ as in (61). It's easy to prove that the system of linear differential equations (65) has a unique solution g (see Perko (1991) Theorem 1). In fact, the equation (65) can be written as

$$\mathbf{g'} = \mathbf{A}\mathbf{g} \tag{66}$$

where $\mathbf{g}(t) := \begin{pmatrix} g(t,1) \\ g(t,2) \\ \dots \\ g(t,N) \end{pmatrix}$ and

$$\mathbf{A} := \begin{pmatrix} q_{11}+a(1) & q_{12} & \vdots & q_{1N} \\ q_{21} & q_{22}+a(2) & \dots & q_{2N} \\ \vdots & \vdots & \dots & \vdots \\ q_{N1} & q_{N2} & \vdots & q_{NN}+a(N) \end{pmatrix}.$$

Thus, the solution to (66) is given by

$$\mathbf{g}(t) = e^{\mathbf{A}t}\mathbf{g}_0, \tag{67}$$

where, the terminal condition $g(T,i) = 1$ for $i \in E$ implies that $\mathbf{g}_0 = e^{-AT}$, so,

$$\mathbf{g}(t) = e^{\mathbf{A}(t-T)}. \tag{68}$$

As a result the function g satisfies the HJB equation (55), see (65). Moreover, the function G given in (62) is in $C^{1,2}([0,T] \times \mathbb{R}^+ \times E)$, and taking $M(t) := \max_{i \in E}|g(t,i)|$, G satisfies the growth condition, i.e., $|\ G(t,x,i)\ | \le K(1+ \mid x \mid)$ for a suitable constant K. Consequently, the desired conclusion follows from Theorem 4.1. □

4.2 Maximizing exponential utility with positive interest rate

In this section we assume that our company may also invest in a risk-free asset which has a positive interest rate $r > 0$. Therefore, in addition to the risky stock, given by (48), and the risk process $Y(t)$ in (49), there is also a bond whose price $B(t)$ evolves as

$$dB(t) = rB(t)dt, \tag{69}$$

with $r > 0$. In this case, any wealth not invested in the stock, say $X^f(t) - f(t)$, will be invested in the bond. Hence, for any strategy f in $\mathcal{A}(t, \widehat{\mathcal{F}})$, the wealth process $X^f(\cdot)$ is now given by

$$\begin{aligned} dX^f(t) &= f(t)\frac{dP(t)}{P(t)} + (X^f(t) - f(t))\frac{dB(t)}{B(t)} + dY(t) \\ &= [rX^f(t) + f(t)(\mu(\psi(t)) - r) + \alpha]dt \\ &\quad + f(t)\sigma(\psi(t))dW_1(t) + \beta dW_2(t). \end{aligned} \tag{70}$$

The generator of the new wealth process is

$$\begin{aligned} &\mathcal{L}_r^f v(t,x,i) = v_t(t,x,i) + [f(t)(\mu(i) - r) + rx + \alpha]v_x(t,x,i) \\ &+ \frac{1}{2}[f(t)^2\sigma^2(i) + \beta^2 + 2\rho\sigma(i)\beta f(t)]v_{xx}(t,x,i) \\ &+ \sum_{j=1}^{N} q_{ij}v(t,x,j) \\ &= \mathcal{L}^f v(t,x,i) + r(x - f(t))v_x(t,x,i). \end{aligned} \tag{71}$$

with $\mathcal{L}^f$ as in (56).

We assume that $\mu(i) > r$ for all $i \in E$. We wish to find an investment strategy that maximizes the terminal utility $\mathbb{E}_{t,x,i}[u(X^f(T))]$. To this end, we use Theorem 4.1 with the HJB-equation (55), but replacing the generator $\mathcal{L}^f v(t,x,i)$ by $\mathcal{L}_r^f v(t,x,i)$ in (71).

Theorem 4.3 *Assume $r > 0$. Consider the Problem (53) for the utility function u given in (57). Then the optimal strategy is to invest the amount*

$$f^*(t,x,i) = \frac{\mu(i) - r}{\theta\sigma^2(i)}e^{-r[T-t]} - \frac{\rho\beta}{\sigma(i)} \tag{72}$$

in the risky stock at time t. Moreover, the value function is given by

$$v(t,x,i) = \lambda - \frac{\gamma}{\theta}g(t,i)e^{-\theta x e^{r[T-t]}}, \tag{73}$$

where $g(t,i)$ satisfies

$$g_t(t,i) + g(t,i)b(i) + \sum_{j=1}^{N} q_{ij}g(t,j) = 0 \ \ \forall i \in E, \tag{74}$$

with boundary condition $g(T,i) = 1$, and

$$\begin{aligned} b(i) &:= xre^{r[T-t]}\theta + \Big[rx + \alpha - \frac{\rho\beta}{\sigma(i)}(\mu(i) - r)\Big]\frac{\theta r}{\gamma}e^{r[T-t]} \\ &+ \frac{1}{2}\Big(\frac{\mu(i) - r}{\sigma(i)}\Big)^2\theta \\ &- \frac{1}{2}\beta^2(1 - \rho^2)\theta^2 e^{2r[T-t]} \quad \text{for all } i \in E. \end{aligned} \tag{75}$$

Proof. Suppose that a solution $G \in C^{1,2}([0,T]\times\mathbb{R}^+\times E)$ of the HJB-equation (55) can be written as

$$G(t,x,i) = \lambda - \frac{\gamma}{\theta}g(t,i)e^{-\theta x e^{r[T-t]}}, \tag{76}$$

for some nonnegative function $g(\cdot,i) \in C^1([0,T],\mathbb{R})$ for all $i \in E$. Then the derivatives of G are given by

$$\begin{aligned} G_t(t,x,i) &= -\frac{\gamma}{\theta}e^{-\theta x e^{r[T-t]}}g_t(t,i) \\ &\quad - [g(t,i)\gamma x r e^{r[T-t]}]e^{-\theta x e^{r[T-t]}}, \\ G_x(t,x,i) &= \gamma e^{r[T-t]}g(t,i)e^{-\theta x e^{r[T-t]}}, \\ G_{xx}(t,x,i) &= -\gamma\theta e^{2r[T-t]}g(t,i)e^{-\theta x e^{r[T-t]}}. \end{aligned} \tag{77}$$

Hence, since $g \geq 0$, the function $x \mapsto G(t,x,i)$ is concave. Moreover, inserting (77) in (55) we see that the maximizer f^* of the HJB-equation (55) is given by

$$f^*(t,x,i) = -\frac{\mu(i) - r}{\sigma^2(i)}\frac{G_x(t,x,i)}{G_{xx}(t,x,i)} - \frac{\rho\beta}{\sigma(i)}. \tag{78}$$

By (77) and (78) it follows that f^* is an admissible portfolio strategy. Replacing the derivatives (77) and f^* in the HJB-equation we obtain the system of equations

$$g_t(t,i) + g(t,i)b(i) + \sum_{j=1}^{N} q_{ij}g(t,j) = 0 \ \ \forall i \in E, \tag{79}$$

with boundary condition $g(T,i) = 1$, and $b(i)$ as in (75). The existence of a solution to (79) is well known. In fact, the equation (79) can be written as

$$\mathbf{g'} = \mathbf{A}\mathbf{g} \tag{80}$$

where $\mathbf{g}(t) := \begin{pmatrix} g(t,1) \\ g(t,2) \\ \dots \\ g(t,N) \end{pmatrix}$ and

$$\mathbf{A} := \begin{pmatrix} q_{11}+b(1) & q_{12} & \vdots & q_{1N} \\ q_{21} & q_{22}+b(2) & \ldots & q_{2N} \\ \vdots & \vdots & \ldots & \vdots \\ q_{N1} & q_{N2} & \vdots & q_{NN}+b(N) \end{pmatrix}.$$

Thus, the solution to (80) is given by

$$\mathbf{g}(t) = e^{\mathbf{A}t}\mathbf{g}_0, \tag{81}$$

where, the terminal condition $g(T,i) = 1$ for $i \in E$ implies that $\mathbf{g}_0 = e^{-AT}$, so,

$$\mathbf{g}(t) = e^{\mathbf{A}(t-T)}. \tag{82}$$

Finally, replacing (77) in (78) gives the optimal strategy (72). Again, as a result, G is in $C^{1,2}([0,T] \times \mathbb{R}^+ \times E)$, and G satisfies the growth condition, i.e., $\mid G(t,x,i) \mid\leq K(1+\mid x \mid)$ for a suitable constant K considering $M(t) := \max_{i\in E}|g(t,i)|$. The optimality of (72) and (73) follows from Theorem 4.1. □

5 Concluding remarks

This paper shows applications of the Itô's stochastic calculus to the problem of modeling a series RLC electrical circuit, liquid level system and, angular velocity of a DC motor, including experimental measurements (in some cases) and both deterministic and stochastic solutions for the ODE and SDEs, respectively. In the optimal control area, an optimal investment problem was solved using the dynamic programming approach. Stochastic differential equations are an interesting alternative in order modeling different processes in engineering, because these processes present randomness and therefore, the SDEs are more appropriate for describing them as it saw in the applications developed in this work. Although in this paper, we focus on the Itô's integral, there is also the Stratonovich's integral, but this is most frequently used within the physical sciences.

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