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Stochastic LQR optimal control with white and colored noise: Dynamic programming technique

Control óptimo estocásticio LQR con ruido blanco y coloreado: Técnica de programación dinámica

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Abstract

This work deals with LQR optimal control problems where the randomness of dynamic systems evolves according to: (a) multiplicative and additive white noise, (b) mixture of white noise and (c) white noise and colored noise. Total cost quadratic and discounted cost quadratic are studied. The white noise is represented by the Wiener process whereas the colored noise is represented by the solution of the Ornstein-Uhlenbeck's stochastic differential equation, with constant initial condition or normally distributed. In order to find the value functions and optimal policies, as well as algebraic and stochastic Riccati equations, the dynamic programming technique was used. The theoretical results are illustrated by four applications. Numerical simulations are carried out using Matlab.

Keywords: Brownian motion, correlation time, Itô 's calculus, Markov processes, stochastic Ricatti equation.

Resumen

Este trabajo trata sobre problemas de control óptimos LQR donde la aleatoridad de los sistemas dinámicos evoluciona de acuerdo a: (a) ruido blanco aditivo y multiplicativo, (b) ruido blanco mezclado y (c) ruido blanco y ruido coloreado. Se estudian el costo cuadrático total y el costo cuadrático descontado. El ruido blanco es representado por el proceso de Wiener mientras que el ruido coloreado es representado por la solución de la ecuación diferencial de Ornstein-Uhlenbeck con condición inicial constante o distribuida normalmente. La técnica de programación dinámica es usada para encontrar las funciones de valor, los controles óptimos y las ecuaciones de Ricatti estocásticas. Los resultados teóricos son ilustrados con cuatro aplicaciones. Las simulaciones numéricas se realizaron en MatLab.

Palabras clave: Movimiento Browniano, tiempo de correlación, Cálculo de Itô, procesos de Markov, ecuación de Ricatti estocástica.

1 Introduction

Many works assume that studied processes $x(t)$ can be described by ordinary differential equations (ODE) that take the form $dx(t) = b(t, x(t), u(t))dt$ for a given function b and a process $u(\cdot)$, and, it is assumed that the process $x(t)$ is unaffected by random perturbations (noises). However, in real systems, every physical process shows perturbations, such disturbances can be added to the processes in an additive or multiplicative way depending on the modelling of the process under study. When the noise is considered to affect the system additively, the ODE is replaced by the stochastic differential equation (SDE) $dx(t) = b(t, x(t), u(t))dt + \xi(t)dt$, where ξ is some stationary stochastic process (a process for which the statistical characteristics do not change with time) with mean zero and known autocovariance matrix $R(\tau) := \mathbb{E}[\xi(t)\xi(t+\tau)]$, the notation $\mathbb{E}$ refers to the expected value operation taken over the ensemble of stochastic processes $\{x(t, \omega)\}$. If the noise affects the process in multiplicative form, then the SDE proposal is $dx(t) = b(t, x(t), u(t))dt + x(t)\xi(t)dt$. Finally, if the noise affects the process in both ways, then one takes as a model the SDE

$$dx(t) = b(t, x(t), u(t))dt + x(t)\xi(t)dt + \xi_1(t)dt, \quad (1)$$

where ξ_1 is another stationary stochastic process which can be correlated to the process ξ.

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It is possible to analyze a stationary stochastic process ξ by its statistic properties (mean, variance, autocovariance, etc.) or its spectral density which is defined as the Fourier transform of the autocovariance function

$$S_{\xi}(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-2\pi j\omega\tau} d\tau.$$

The relationship between the spectral density function $S_{\xi}(\omega)$ and autocovariance function $R(\tau)$, which is summarised in the Wiener-Khintchine theorem, provides a link between the time-domain and the frequency domain analyses from the stationary stochastic processes, Kasdin (1995).

A stationary white Gaussian noise process is defined as a zero-mean process $\mathbb{E}[\xi(t)] = 0$ with infinite variance for which, $R(\tau) = \delta(\tau)$, where $\delta(\cdot)$ is the Dirac delta ($\delta(\tau) = 0$ if $\tau \neq 0$ and $\delta(\tau) = 1$ for $\tau = 0$). The stationary white Gaussian noise has spectral density constant $S_{\xi}(\omega) = \frac{1}{2\pi}$ for $-\infty < \omega < \infty$, for this reason, It is called *the white noise* and the process contains equal information at all frequencies. In 1908, Paul Langevin (French physicist) found that if the Brownian motion $W(t)$ were differentiable, then its derivative would be the white Gaussian noise process, i.e., $dW(t) = \xi(t)dt$. Therefore, in this case, replacing $\xi(t)dt$ by $dW(t)$ in (1), we get

$$dx(t) = b(t, x(t), u(t))dt + x(t)dW(t) + \sigma dW_1(t), \quad (2)$$

where σ is a constant matrix known as the intensity of noise. Thus, (2) it is a first approximation to (1) (real dynamics).

In this work, in addition to the stationary white Gaussian process a coloured Gaussian noise $w(t)$ is considered. The colored noise process is a zero-mean stochastic process, exponential correlation function and whose spectral density is not constant for all ω, Jung et al. (2005). In this case, the process $w(t)$ can be regarded as the Ornstein-Uhlenbeck process. As mentioned in Jia and Li (1998), there are many systems in physics and other sciences, which are driven simultaneously by white and colored noise sources; for instance, the laser systems, the lattice model, the structure-formation process in liquid crystals, etc. Here, in addition of the SDE (2) we are interested in SDEs driven by stationary white Gaussian noise and a colored Gaussian noise of the form

$$dx(t) = b(t, x(t), w(t), u(t))dt + \sigma(x(t), w(t))dW(t), \quad (3)$$

and

$$dx(t) = b(t, x(t), u(t))dt + \sigma(x(t))dW(t) + dw(t). \quad (4)$$

The Itô's theory of stochastic calculus allows us to study systems affected by noises (perturbations) because the ODEs associated with the systems studied are replaced by SDEs. In Engineering, stochastic calculus is used in filtering and control theory. As well as to study the effects of random excitation on various physical phenomena and to model the effects of stochastic variability in reproduction and environment on populations in Physics and Biology, respectively, Klebaner (2005). In recent years the study of SDEs to model physical systems increased, some important applications are: *Stochastic analysis of the power output for a wind turbine* Anahua et al. (2004). *Stochastic Navier-Stokes equations (SNSE) for turbulent flows* Mikulevicius and Rozovskii (2004) among others.

It is well known that the stochastic linear quadratic regulator (LQR) optimal control problems deal with minimizing/ maximizing a quadratic cost/ reward subject to the dynamic systems evolves according to linear-SDEs. The SDEs (2), (3) and (4) are called *controlled-linear-SDEs* when b and σ are linear functions of x. There are enough works on LQR-optimal control both deterministic and stochastic due to its multiple applications in different areas of science, Kumar and Jain (2019); Prasad et al. (2011); Chen et al. (1998); Kalman (1960); Lewis et al. (1986); Wonham (1968). Moreover, LQR optimal control is used in situations where no linear dynamic becomes linear around fix points through Taylor expansion, Tang et al. (2012); Prasad et al. (2011). Some applications can be found in Hernández-Osorio et al. (2019); Rojas et al. (2016); Razmjooy et al. (2016); López et al. (2011); Sam et al. (2000); Seekhao et al. (2020); Ahmad et al. (2020)

1.1 Our approach and main contributions

The main aim of this work consist of studying the LQR-optimal control problems using dynamic programming (DP) approach where the dynamic systems evolves according to (2), (3) and (4) considering that the function b and σ are linear functions of x. The idea of this approach is to obtain the so-called Hamilton-Jacobi-Bellman equation or dynamic programming equation (DPE) from which we can obtain, under appropriate conditions, the optimal control problem's value function and also optimal control policies. As far as our knowledge goes, our work is the first of this type to study LQR optimal control considering both colored noise and white noise.

2 Linear stochastic differential equations

There are excellent books which give a detailed description of the Itô's stochastic calculus, recommended readers are, Klebaner (2005); Arnold (2013); Durrett (1996); Karatzas et al. (1991).

Consider a n-dimensional controlled stochastic differential equation evolving according to

$$dx(t) = b(t, x(t), u(t))\,dt + \sigma(x(t))dW(t), \qquad (5)$$
$$x(0) = x_0,$$

where $b : [0,\infty) \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n_1}$ are given measurable functions called the *drift* and the *diffusion term*, and $W(\cdot)$ is an n_1 -dimensional standard Brownian motion. The stochastic process $u(\cdot)$ is a U-valued process called a control process, and the set $U \subset \mathbb{R}^m$ is called the control (or action) space. The linear growth condition and Lipschitz condition on b and σ ensures the existence of a unique continuous strong solution $x(\cdot)$, which is a Markov process. Moreover, denoting by $\mathbb{E}^u_{s,x}$ the conditional expectation given initial state x and the sequence controllers u, we also have

$$\mathbb{E}^u_{s,x}|x(t)|^k \leq (1 + |x|^k)e^{C(t-s)} \quad k = 1, 2, \ldots \qquad (6)$$

for some constant C depending on the integer k and the constant $K(T)$, see Klebaner (2005); Durrett (1996); Pham (2009); Arnold (2013).

This paper deals with linear stochastic differential equations, i.e., the functions b and σ in (5) are linear functions of $x \in \mathbb{R}^n$. More precisely, we are interested in,

- *Linear SDE driven by two Brownian motions*

  $$dx(t) = (Ax(t)+au(t))dt+Bx(t)\,dW_1(t)+bdW_2(t). \qquad (7)$$

  The Brownian motions $W_1(\cdot)$ and $W_2(\cdot)$ can be correlated, with correlation coefficient $|\gamma| < 1$, that is, $\mathbb{E}[W_1(t)W_2(t)] = \gamma t$. Here, A, a, B and b are $(n\times n)$, $(n\times m)$, $(n\times n_1)$ and $(n\times n_2)-$ matrix-valued, respectively. In addition, W_1 and W_2 are n_1 and n_2 -dimensional standard Brownian motions, respectively.

- *Linear SDE driven by a mixture noise, Baloui Jamkhaneh (2011).* A mixture noise may be interpreted as a linear combination of n independent Brownian motions W_i, $i = 1, 2, \ldots n$, that is

  $$d\phi(t) = \sum_{i=1}^{n} \alpha_i dW_i(t),$$

  where α_i are constants satisfying $\sum_{i=1}^n \alpha_i = 1$. Hence, linear SDE driven by a mixture noise is written as

  $$dx(t) = (Ax(t) + au(t))dt + d\phi(t). \qquad (8)$$

- *Linear SDE driven by a white noise and a colored noise.* The colored noise evolves according to the linear stochastic differential equation

  $$dw(t) = -\rho w(t)dt + \sigma\rho d\overline{W}(t), \qquad (9)$$

  where $\sigma, \rho > 0$ are fixed constants, $\overline{W}(t)$ is an one-dimensional Brownian motion and $w(0) \sim N(0, \sigma^2/2\rho)$ (random variable with normal distribution, zero mean and variance $\sigma^2/2\rho$), see (Arnold, 2013, section 8.3) for more details on colored noise. Depending on the way the colored noise is added to the system, the linear SDE driven by a white noise and a colored noise can have the following structures.

  $$\begin{aligned} dx(t) &= (A(w(t))x(t) + a(w(t))u(t))dt \\ &+ b(w(t))dW(t), \end{aligned} \qquad (10)$$

  or

  $$dx(t) = (Ax(t) + au(t))dt + bdw(t). \qquad (11)$$

  In (10), $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $a : \mathbb{R} \to \mathbb{R}^{n\times m}$, $b : \mathbb{R} \to \mathbb{R}^{n\times n_1}$ and W is a n_1 -dimensional standard Brownian motion independent of $\overline{W}$, whereas, in (11), the matrix dimensions A, a and b are given in (7).

2.1 *Itôs calculus.*

We now introduce some notation which will be used throughout. Let $C^{1,2,2}([0,T] \times \mathbb{R}^n \times \mathbb{R})$ be the space of real-valued functions $h(t, x, w)$ on $[0,T] \times \mathbb{R}^n \times \mathbb{R}$ which are once differentiable in t and twice continuously differentiable in x and w. Similarly, we define the space $C^{1,2}([0,T]\times\mathbb{R}^n)$. For each $h(t, x, w) \in C^{1,2,2}([0,T] \times \mathbb{R}^n \times \mathbb{R})$, we denote by h_x and h_{xx} the gradient (row) vector and the Hessian matrix of h, respectively. The following lemma shows the Itô's lemma (also known as the fundamental theorem of the stochastic calculus). For a proof we quote (Friedman, 2012, Theorem 5.3) or (Morimoto, 2010, Theorem 1.6.2).

Lemma 2.1. *Itô's lemma.*

(*a*) *Let $x(\cdot)$ be a solution of the SDE (5) and let $h(t,x) \in C^{1,2}([0,T]\times\mathbb{R}^n)$. Then, the stochastic process $y(t) = h(t,x(t))$ satisfies the stochastic differential equation*

$$\begin{aligned} dy(t) &= [h_t(t,x(t))dt + b(t,x(t))h_x(t,x(t)) \\ &+ \frac{1}{2}h_{xx}(t,x(t))]dt + h_x(t,x(t))\sigma(x(t))dW(t) \end{aligned} \quad (12)$$

(*b*) *Let $x(\cdot)$ be a solution of the stochastic differential equation with two Brownian motions affecting the system in multiplicative and additive form, as follows.*

$$\begin{aligned} dx(t) &= b(t,x,u(t))dt + \sigma_1(x(t))dW_1(t) \\ &+ \sigma_2(x(t))dW_2(t) \end{aligned} \quad (13)$$

with $\mathbb{E}[W_1(t),W_2(t)] = \gamma t$, $\sigma_2(x(t)) = \sigma x(t)$ and $\sigma,\gamma > 0$. Now, take $h(t,x) \in C^{1,2}([0,T]\times\mathbb{R}^n)$, the stochastic process $y(t) = h(t,x(t))$ satisfies the stochastic differential equation

$$\begin{aligned} dy(t) &= [h_t(t,x(t))dt + b(t,x(t))h_x(t,x(t)) \\ &+ \frac{1}{2}h_{xx}(t,x(t))[Tr[\sigma_1(x(t))\sigma_1(x(t))^T] \\ &+ Tr[\sigma_2(x(t))\sigma_2^T(x(t))]] \\ &+ 2\sigma_1(x(t))\sigma_2(x(t))\gamma]dt \\ &+ h_x(t,x(t))\sigma_1(x(t))dW_1(t) \\ &+ h_x(t,x(t))\sigma_2(x(t))dW_2(t) \end{aligned} \quad (14)$$

(*c*) *Let $x(\cdot)$ be a solution of the controlled SDE (5) and let $w(\cdot)$ be the solution of stochastic differential equation*

$$dw(t) = b_1(t,w(t))dt + \sigma_3(w(t))dW_3(t) \quad (15)$$

where $\mathbb{E}[W(t),W_3(t))] = \gamma t$ and $\gamma > 0$. For some $h(t,x,w) \in C^{1,2,2}([0,T]\times\mathbb{R}^n\times\mathbb{R})$, the stochastic process $y(t) = h(t,x(t),w(t))$ satisfies the stochastic differential equation

$$\begin{aligned} dy(t) &= [h_t(t,x(t),w(t)) \\ &+ b_1(t,w(t))h_w(t,x(t),w(t)) \\ &+ b(t,x(t))h_x(t,x(t),w(t)) \\ &+ \frac{1}{2}h_{xx}(t,x(t),w(t))Tr[\sigma(x(t))\sigma^T(x(t))] \\ &+ \frac{1}{2}h_{ww}(t,x(t),w(t))Tr[\sigma_3(w(t))\sigma_3^T(w(t))] \\ &+ \gamma\sigma_3(w(t))\sigma(x(t))h_{xw}(t,x(t),w(t))]dt \\ &+ h_x(t,x(t),w(t))\sigma(x(t))dW(t) \\ &+ h_w(t,x(t),w(t))\sigma_3(w(t))dW_3(t) \end{aligned} \quad (16)$$

The stochastic differential equations (12), (14) and (16) are called the Itô's formulas and are an extension of the stochastic theory of the chain rule of ordinary calculus.

3 LQR optimal control

In this work, we are interested in LQR optimal control problems with finite and infinite-horizon. The cost that an agent obtains from their activity in the system is measured with the performance indexes following.

$$J_T(t,x,u) := \mathbb{E}_x^u\left[\int_t^T r(s,x(s),u(s))\,\mathrm{d}s + r_1(T,x(T))\right], \quad (17)$$

and

$$J(x,u) := \mathbb{E}_x^u\left[\int_0^\infty e^{-\alpha s}r(s,x(s),u(s))\,\mathrm{d}s\right], \quad (18)$$

where $t \in [0,T]$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and the discount factor $\alpha > 0$. Moreover, $r(s,x(s),u(s)) := x^T(s)Qx(s) + u^TRu(s)$ and $r_1(T,x(T)) := x^T(T)Fx(T)$ are the running and terminal cost, respectively, and F,Q and R are positive semi-definitive symmetric matrices with $F,Q \in \mathbb{R}^{n\times m}$ and $R \in \mathbb{R}^{m\times m}$. The control process $u(\cdot)$ depends on the information available to the controller. Here, to simplify the presentation, we only consider Markov (or feedback or closed-loop) controls which are defined as follows.

Definition 3.1. *Let $\mathbb{M}$ be the family of measurable functions $f : [0,\infty)\times\mathbb{R}^n \to U$, and $\mathbb{F}\subset\mathbb{M}$ the subfamily of functions $f : \mathbb{R}^n \to U$. A control policy of the form $u(t) := f(t,x(t))$ for some $f \in \mathbb{M}$ is called a* Markov policy*, whereas $u(t) := f(x(t))$ for some $f \in \mathbb{F}$ is said to be a* stationary Markov policy.

Definition 3.2. *(LQR optimal control). Let $J(t,x,u)$ and $J(x,u)$ be as in (17) and (18), respectively. We say that a policy $f^* \in \mathbb{M}$ (or $\mathbb{F}$) is optimal for the LQR optimal control problems if*

$$J^*(t,x) = \min_{u\in U} J(t,x,u) = J(t,x,f^*) \quad (19)$$

or

$$J^*(x) = \min_{u\in U} J(x,u) = J(x,f^*) \quad (20)$$

subject to that $x(t)$ is a solution of the stochastic differential equations (7) or (8) or (10). The functions $J^*(t,x)$ and $J^*(x)$ are known as the value functions.

4 Dynamic programming approach

We introduce the dynamic programming approach to study the LQR optimal control problem. The dynamic programming (DP) technique is based on the *principle of optimality* stated by Richard Bellman (1920-1984) and gives sufficient conditions for the existence of an optimal control policy, see Durrett (1996); Pham (2009). The DP approach is as follows.

1. First, for each optimal control problem, the DP equation (also known as Hamilton-Jacobi-Bellman equation) must be deduced;
2. next, it obtains or tries to show the existence of a smooth solution (i.e., $v \in C^{1,2}([0,\infty) \times \mathbb{R}^n)$) of PD-equation by partial differential equations techniques;
3. then, it shows that the smooth solution is the value function from optimal problem control by Dynkin's formula, and
4. finally, as a byproduct, it obtains an optimal feedback control.

Of fundamental importance in the Hamilton-Jacobi-Bellman equation is the infinitesimal generator of the process $x(t)$ governed by the general SDE (5). Next, we calculate the infinitesimal generator of the diffusion process governed by the SDEs (7), (8) and (10). For this purpose, we use the following definition.

Definition 4.1. *The infinitesimal generator of the diffusion process $x(t)$ is the operator $\mathcal{A}$, which is defined to act on functions $h : \mathbb{R}^n \to \mathbb{R}$ by*

$$\mathcal{A}h(x) = \lim_{t\to 0^+} \frac{\mathbb{E}_x[h(x(t)] - h(x)}{t}.$$

Its domain $\mathcal{D}(\mathcal{A})$ is the set of functions h twice differentiable with continuous second derivative and for which the limit exists.

Consider the general controlled stochastic differential equation (5) but driven also by a stochastic process $w(t)$ which is a solution to the SDE (15). This is,

$$dx(t) = b(t, x(t), w(t), u(t))\,dt + \sigma(x(t))dW(t), \quad (21)$$
$$x(0) = x_0, \;\; w(0) = w_0.$$

Now, define the operator $\mathcal{L}^f : C^{1,2,2}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}) \to C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R})$ as

$$\begin{aligned}\mathcal{L}^f h(t,x,w) &:= h_t(t, x(t), w(t)) \\ &+ b_1(t, w(t))\, h_w(t, x(t), w(t)) \\ &+ b(t, x(t), u(t))\, h_x(t, x(t), w(t)) \\ &+ \frac{1}{2} h_{xx}(t, x(t), w(t))\, Tr[\sigma(x(t))\sigma(x(t))^T] \\ &+ \frac{1}{2} h_{ww}(t, x(t), w(t))\, Tr[\sigma_3(w(t))\sigma_3^T(w(t))] \\ &+ \gamma\sigma_3(w(t))\sigma(x(t)) h_{xw}(t, x(t), w(t)) \qquad (22)\end{aligned}$$

Integrating the Itô's formula (16) from s to t and taking expected value, we have

$$\mathbb{E}_{x,w}[h(t, x(t), w(t))] - h(s,x,w) = \mathbb{E}_{x,w}\Big[\int_s^t \mathcal{L}^f h(s, x(s), w(s)) ds\Big] \quad (23)$$

since the expected value of the stochastic integrals are zero, Klebaner (2005); Durrett (1996), that is

$$\mathbb{E}_{x,w}\Big[\int_s^t h_x(s, x(s), w(s))\, \sigma(x(s))\, dW(s)\, ds\Big] = 0,$$

$$\mathbb{E}_{x,w}\Big[\int_s^t h_w(s, x(s), w(s))\, \sigma_3(w(s))\, dW_3(t)\, ds\Big] = 0.$$

Using Fubini's theorem and the Fundamental theorem of calculus, we get

$$\begin{aligned}\mathcal{A}h(s,x,w) &= \lim_{t\to s} \frac{\mathbb{E}_x[h(t, x(t), w(t))] - h(s,x,w)}{t-s} \\ &= \lim_{t\to s} \frac{1}{t-s} \int_s^t \mathbb{E}_{x,w}[\mathcal{L}^f h(s, x(s), w(s))] ds \\ &= \mathbb{E}_{x,w}[\mathcal{L}^f h(s,x,w)] \\ &= \mathcal{L}^f h(s,x,w).\end{aligned}$$

So, this last result implies that the infinitesimal generator from the process (21) coincides with the operator (22). Similar arguments to those given above allow us to calculate the infinitesimal generator of the processes (7), (8) and (10).

- If $x(t)$ is governed by the linear SDE driven by two independent Brownian motions (7) with $\gamma = 0$, then, its infinitesimal generator $\mathcal{L}^f$ applied to $v(t,x) \in C^{1,2}([0,T] \times \mathbb{R}^n)$ is

$$\begin{aligned}\mathcal{L}^f v(t,x) &= v_t(t,x) + (Ax + au)v_x(t,x) \\ &+ \frac{1}{2}[Tr(bb^T) + Tr(Bx(Bx)^T)]v_{xx}(t,x) \qquad (24)\end{aligned}$$

- Assuming the $x(t)$ evolves according to the linear SDE driven by a mixture noise (8), its infinitesimal generator $\mathcal{L}^f$ applied to $v(t,x) \in C^{1,2}([0,T]\times\mathbb{R}^n)$ is given by

$$\begin{aligned}\mathcal{L}^f v(t,x) &= v_t(t,x) + (Ax+au)v_x(t,x)\\ &+ \frac{1}{2}\sum_{i=1}^{n}\alpha_i^2 v_{xx}(t,x)\end{aligned} \quad (25)$$

- For the linear SDE driven by a white noise and a colored noise (10) with $\gamma = 0$, the infinitesimal generator $\mathcal{L}^f$ of the process $x(t)$ applied to $v(t,x,w) \in C^{1,2,2}([0,T]\times\mathbb{R}^n\times\mathbb{R})$ is

$$\begin{aligned}\mathcal{L}^f v(t,x,w) &= v_t(t,x(t),w) - \rho w(t)v_w(t,x,w)\\ &+(A(w(t))x(t) + a(w(t))u(t))v_x(t,x,w)\\ &+\frac{1}{2}Tr(b(w(t))b^T(w(t)))v_{xx}(t,x,w)\\ &+\frac{1}{2}\sigma^2\rho^2 v_{ww}(t,x,w).\end{aligned} \quad (26)$$

HJB equations for the LQR optimal problems.

The Hamilton-Jacobi-Bellman (HJB) equations associated with the finite-horizon quadratic cost (17) and discounted quadratic cost (18) are

$$\min_{u\in U}\{x(t)Qx(t)^T + u(t)^T Ru(t) + \mathcal{L}^u v(t,x)\} = 0, \quad (27)$$

subject to the terminal condition $x^T(T)Fx(T) = v(T,x(T))$, and

$$\alpha v(x) = \min_{u\in U}\{x(t)Qx^T(t) + u^T(t)Ru(t) + \mathcal{L}^u v(x)\}, \quad (28)$$

respectively. The infinitesimal generator $\mathcal{L}^u$ in the equations (27) and (28) depends on the evolution of diffusion process $x(t)$, i.e., if $x(t)$ is governed by the SDE (7), then $\mathcal{L}^u$ is given by the operator (24).

Remark 4.2.

(a) (Vanishing discount technique). Another index basic for LQR is the long-run expected average cost per unit time *which is defined as*

$$J(x,u) := \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}_x^u\Big[\int_0^T [x^T(t)Qx(t)+u^T(t)Ru(t)]dt\Big].$$

The value function for this cost is $J^*(x) := \min_{u\in U} J(x,u)$. *It is well known that if there is a pair* $(j^*,h)\in\mathbb{R}\times C^2(\mathbb{R}^n)$ *and a policy* $u^*\in U$ *satisfying the average HJB-equation*

$$\begin{aligned}j^* &= \min_{u\in U}\{x^T(t)Qx(t) + u^T(t)Ru(t) + \mathcal{L}^u h(x)\} \quad (29)\\ &= x^T(t)Qx(t) + u^{*T}(t)Ru^*(t) + \mathcal{L}^{u^*}h(x),\end{aligned}$$

then, $j^* = J^*(x)$ *and* u^* *is an optimal policy. There are several ways to get* (j^*,h)*, the most common being those based on variants on the* vanishing discount technique*, in which we obtain the HJB-equation for average control problem by letting* α *tend to zero in a class of* α*-discounted cost problems. The basic idea is as follows. We first consider the* $\alpha-$ *discount dynamic programming equation (28); next, select an arbitrary state* $x^*\in\mathbb{R}^n$ *and define* $j(\alpha) := \alpha v(x^*)$ *and* $h_\alpha(x) := v(x) - v(x^*)$. *Thus (28) can also be written as*

$$\alpha h_\alpha(x) + j(\alpha) = \min_{u\in U}\{x^T(t)Qx(t)+u^T(t)Ru(t)+\mathcal{L}^u h_\alpha(x)\}. \quad (30)$$

Comparison of (30) with (29) immediately suggest to let α *tend to zero in (30) to obtain (29) in the limit. In fact,* $j(\alpha)\to j^*$, $\alpha h_\alpha(x)\to 0$ *and* $h_\alpha(x)\to h(x)$ *as* $\alpha\to 0$. *See (Hernández-Lerma, 1994, pages 45-49) for more details.*

Smooth solution of HJB equations.

A consequence of the following *Verification Theorems*, whose proofs are quite standard (see, for instance, Theorem 3.5.2 and Theorem 3.5.2 in Pham (2009)) is that the pair $(J^*(t,x), f^*)$ consisting of the value function $J^*\in C^2([0,T]\times\mathbb{R}^n)$ and a policy $f^*\in\mathbb{F}$ is a solution of the HJB equation (27).

Theorem 4.3. (*Finite horizon*) *Suppose that* $G \in C^{1,2}([0,T]\times\mathbb{R}^n)$ *is a solution of the HJB equation* (27) *that satisfies the growth condition* $|G(t,x)| < K(1+|x|^k)$ *for some constants* $K > 0$ *and* $k \geq 1$, *and* $0 \leq t \leq T$. *Then*

(*a*) $G(t,x) \geq J^*(t,x)$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^n$.

(*b*) *Let* $u(t) := f^*(t,x)$ *be the minimizer of HJB equation* (27). *Then* $G(t,x) = J^*(t,x)$ *on* $[0,T]\times\mathbb{R}^n$. *Moreover,* $f^*\in\mathbb{F}$ *is an optimal Markovian policy.*

Theorem 4.4. (*Infinite horizon*) *Suppose that* $G \in C^2(\mathbb{R}^n)$ *is a solution of the HJB-equation* (28) *that satisfies the growth condition* $|G(x)| < K(1 + |x|^k)$ *for some constants* $K > 0$ *and* $k \geq 1$. *Then*

(*a*) $G(x) \geq J^*(x)$ *on* $\mathbb{R}^n$ *for every policy* $f\in\mathbb{F}$ *such that*

$$e^{-\alpha t}\mathbb{E}_x^f[G(x(t))] \to 0 \ \text{as } t\to\infty. \quad (31)$$

(*b*) *Let* $u(t) := f^*(x)$ *be the minimizer of HJB equation* (28). *Then* $G(x) = J^*(x)$ *on* $\mathbb{R}^n$. *Moreover,* $f^*\in\mathbb{F}$ *is an* $\alpha-$*discount optimal Markovian policy in the class of policies* $f\in\mathbb{F}$ *that satisfy (31).*

5 Main results

Theorems 4.3 and 4.4 are applied in the proof from Propositions 5.1 and 5.2. These propositions give the explicit forms of both the value functions and optimal policies that solves the LQR stochastic optimal control problems (LQR-OCPs) when the dynamic systems evolve according to (10).

Proposition 5.1. *(Finite horizon LQR-OCP). Assume that $x(t)$ evolves according to an SDE driven by white noise and colored noise as in (10). Then, the policy that minimizes the finite cost (17) at each time $0 \leq t \leq T$, is*

$$f^*(t,x,w) = -R^{-1}a^T(w)K(t)x(t), \qquad (32)$$

whereas that the corresponding value function is given by

$$v(t,x,w) = x^T(t)K(t)x(t) + g(w(t)), \qquad (33)$$

where $K(t)$ is a positive semi-definite matrix that satisfies the Ricatti differential equation

$$\begin{aligned} K'(t) + A^T(w)K(t) + K(t)A(w) \\ -K(t)a(w)R^{-1}a^T(w)K(t) \\ +Q + Tr[b(w)b^T(w)]K(t) = 0, \end{aligned} \qquad (34)$$

and $g(w(t))$ satisfies the ordinary differential equation

$$-\rho w g'(w) + \frac{1}{2}\sigma^2\rho^2 g''(w) = 0. \qquad (35)$$

Proof. The HJB equation for the LQR optimal control problem (20) with $x(t)$ evolves according to (10) and finite cost (17) is

$$\min_{u\in U}\{xQx^T + u^T Ru + \mathcal{L}^u v(t,x,w)\} = 0, \qquad (36)$$

together with the terminal condition $x^T(T)Fx(T) = v(T,x(T),w(T))$, where $\mathcal{L}^u v(t,x,w)$ is the infinitesimal generator given in (26). We are looking for a candidate solution $v \in C^{1,2}([0,T]\times\mathbb{R}^n\times\mathbb{R})$ to (36) in the form

$$v(t,x,w) = x^T K(t)x + g(w), \qquad (37)$$

for some continuous function $g(\cdot)$ and $K(\cdot)$ a positive semi-definite matrix. We assume that $g'' > 0$ for all $w \in \mathbb{R}$, so that, the function $(x,w) \to v(t,x,w)$ is convex. In addition, note that

$$\begin{aligned} |v(t,x,w)| &\leq |x^T(t)|K(t)|x(t)| + |g(w(t))| \\ &\leq \max_{t\in[0,T]}\{K(t), g(w(t))\}(1+|x(t)|^2), \end{aligned}$$

thus, (37) satisfies the growth condition given in Theorem 4.3. Moreover, in order for (36) together with its terminal condition to hold, we required that $K(T) = F$ and $g(w(T)) = 0$.

Now, inserting the partial derivatives of v with respect to x and w in the HJB equation (36), we obtain

$$\begin{aligned} 0 &= x(t)Qx(t)^T + x^T(t)K'(t)x(t) - \rho w(t)g'(w(t)) \\ &+ 2A(w(t))x(t)K(t)x^T(t) \\ &+ Tr(b(w(t))b^T(w(t)))K(t) + \frac{1}{2}\sigma^2\rho^2 g''(w(t))\} \\ &+ \min_{u\in U}\{u^T(t)Ru(t) + 2a(w(t))u(t)x^T(t)K(t)\}. \end{aligned} \qquad (38)$$

Furthermore, given that R is a positive-definite symmetric matrix, the function $u \in U \to u^T Ru + 2a(w)ux^T K$ is strictly convex on the compact set U, and thus, attains its minimum at

$$u(t) = f^*(t,x,w) = -R^{-1}a^T(w(t))K(t)x(t). \qquad (39)$$

Substituting (39) into (38), turns out that $K(\cdot)$ and $g(\cdot)$ should satisfy the Ricatti differential equation (34) and ordinary differential equation (35), respectively. Finally, from the Theorem 4.3 it follows that f^* is an optimal Markovian policy and the value function $J^*(t,x,w)$ is equal to (37). That is,

$$J^*(t,x,w) = \min_{u\in U} J(t,x,u,w) = v(t,x,w) = x^T K(t)x + g(w).$$

□

Proposition 5.2. *($\alpha-$discounted LQR-OCP). Assume that $x(t)$ is governed by an SDE driven by white noise and colored noise, (10). Then, the policy that minimizes the $\alpha-$ discounted cost (18) is*

$$f^*(x(t),w(t)) = -R^{-1}a(w)^T K^T x(t), \qquad (40)$$

and the value function is given by

$$v(x,w) = x^T(t)Kx(t) + g(w), \qquad (41)$$

where K satisfies the algebraic Riccati equation

$$\begin{aligned} Q - Ka(w)(R^{-1})^T a^T(w)K + KA(w) + A(w)^T K \\ +Tr[b(w)b(w)^T]K - \alpha K = 0, \end{aligned} \qquad (42)$$

and $g(w(t))$ satisfies the ordinary differential equation

$$-\rho w g'(w) + \frac{1}{2}\sigma^2\rho^2 g''(w) - \alpha g(w) = 0. \qquad (43)$$

Proof. The HJB equation for the $\alpha-$discounted LQR-OCP is

$$\alpha v(x,w) = \min_{u\in U}\{xQx^T + u^T Ru + \mathcal{L}^u v(x,w)\} = 0. \quad (44)$$

with $\mathcal{L}^u v(x,w)$ as in (26) taking $v_t \equiv 0$. To solve (44), we propose a function $v \in C^{2,2}(\mathbb{R}^2 \times \mathbb{R})$ of the form

$$v(x,w) = x^T Kx + g(w) \quad (45)$$

with K a positive- definite symmetric matrix, and g a continuous function, both to be determined. Observe that

$$|v(x,w)| \leq |x^T|K|x| + |g(w)| \leq \max_{w\in\mathbb{R}}\{K, g(w))\}(1+|x|^2),$$

and therefore, the function (45) satisfies the growth condition from Theorem 4.4.

By using similar arguments to those given in the proof from Proposition 5.1, the optimal control is obtained. In fact,

$$f^*(x(t),w(t)) = -R^{-1}a(w)^T K^T x(t). \quad (46)$$

Substituting the derivatives of v and (46) in (44), turns out that K and g should satisfy the algebraic Riccati equation (42) and the ordinary differential equation (43), respectively.

On the other hand, note that the process $x(t)$ associated to the control $f^*(x(t),w(t))$ is $dx(t) = A(w)x(t) - a(w)R^{-1}a^T(w)Kx(t)dt + \sigma dW(t)$. By rearranging terms we obtain

$$dx(t) = -[-A(w) + a(w)R^{-1}a^T(w)K]x(t)dt + \sigma dW(t), \quad (47)$$

which is the so-called Langevin equation. Thus, the solution of the Langevin equation (47) is

$$\begin{aligned} x(t) &= xe^{-[-A(w)+a(w)R^{-1}a^T(w)K]t} \\ &+ \sigma\int_0^t e^{-[-A(w)+a(w)R^{-1}a^T(w)K](t-s)}dW(s). \end{aligned}$$

See, for instance, (Arnold, 2013, Section 8.3). Therefore, by the properties of stochastic integrals,

$$\mathbb{E}_x^{f^*}[e^{-\alpha t}x^T(t)x(t)] =$$

$$\left[x^2 - \frac{\sigma^2}{2[-A(w)+a(w)R^{-1}a^T(w)K]}\right]e^{-(\alpha+2[-A(w)+a(w)R^{-1}a^T(w)K])t}$$

$$+\frac{\sigma^2}{2[-A(w)+a(w)R^{-1}a^T(w)K]}e^{-\alpha t}.$$

Finally, choosing R such as $\alpha + 2[-A(w) + a(w)R^{-1}a^T(w)K] > 0$, we get

$$\lim_{t\to\infty}\mathbb{E}_x^{f^*}[e^{-\alpha t}x^T(t)kx(t)] = 0 \Rightarrow$$
$$\lim_{t\to\infty}\mathbb{E}_x^{f^*}[e^{-\alpha t}v(x(t),w(t))] = 0.$$

Therefore, from the verification Theorem 4.4 it follows that f^* minimizes (44) within the class of admissible stationary policies $\mathbb{F}$ satisfying (31) and the value function $J^*(x,w)$ is equal to (45). □

A similar proof of those gives in Propositions 5.1 and 5.2 allows us to get the Ricatti equations for the LQR optimal control problems with dynamic systems evolving according to the linear SDEs (7) and (8).

Remark 5.3. *Linear SDE driven by two independent Brownian motions.*

- Finite-horizon quadratic cost. *The policy that minimizes the finite cost (17) at each time $0 \leq t \leq T$ have the structure (32) with $x(t)$ solution of (7). The corresponding value function is (33) taking $g(w(t)) = 0$, where $K(t)$ satisfies the Ricatti differential equation*

$$\begin{aligned} &K'(t) + A^T K(t) + K(t)A - K(t)aR^{-1}a^T K(t) + Q \\ &\quad +[Tr(bb^T) + Tr(Bx(t)(Bx(t))^T)]K(t) = 0. \end{aligned}$$

- $\alpha-$discounted quadratic cost. *The policy that minimizes the $\alpha-$ discounted cost (18) is (32) taking $x(t)$ as a solution of (7). The corresponding value function is (33) taking $g(w(t)) = g$ with g a constant and $K(t) = K$ satisfies the algebraic Ricatti equation*

$$\begin{aligned} &Q + A^T K + KA - KaR^{-1}a^T K \\ &+Tr(Bx(t)(Bx(t))^T)K - \alpha K = 0, \end{aligned}$$

and g is the constant

$$g = \frac{Tr(bb^T)K}{\alpha}.$$

Remark 5.4. *SDE driven by mixture noise.*

- Finite-horizon quadratic cost. *The policy that minimizes the finite cost (17) at each time $0 \leq t \leq T$ have the structure (32) with $x(t)$ solution of (8). The corresponding value function is (33) taking $g(w(t)) = 0$, where $K(t)$ satisfies the following Ricatti differential equation*

$$K'(t) + Q + A^T K(t) + K(t)A - K(t)aR^{-1}a^T K(t)$$

$$+\sum_{i=1}^n \alpha_i^2 K(t) = 0. \quad (48)$$

- $\alpha-$discounted quadratic cost. *The policy that minimizes the $\alpha-$ discounted cost (18) is (32) with $x(t)$ solution of (8). The corresponding value function is (33) taking $g(w(t)) = g$ with g a constant and $K(t) = K$ satisfies the algebraic Ricatti equation*

$$Q + A^T K + KA - KaR^{-1}a^T K - \alpha K = 0,$$

$$g = \frac{\sum_{i=1}^{n} \alpha_i^2 K}{\alpha}.$$

6 Applications

In this section four applications are presented with the aim of to illustrate the theory developed in previous sections.

6.1 DC motor

The motor model has electrical variables that are: supply voltage of the rotor, V_{in} current i that will circulate through the rotor (armature current), winding resistance of the rotor R, and the inductance of the winding of the rotor L, the mechanical characteristics are: the angular position θ, angular velocity of rotation of the rotor $\frac{d\theta}{dt}$ and moment of inertia of the rotor shaft J as shown in Fig. 1, see Ruderman et al. (2008) ,Saisudha et al. (2016), Emhemed and Mamat (2012).

The second-order differential equation that model the angular position and angular velocity in a DC motor is given by

$$\frac{d^2\theta}{dt^2} = -\frac{K_b K_T}{RJ}\frac{d\theta}{dt} + \frac{K_T V_{in}}{RJ}, \qquad (49)$$

assuming that in many motors the inductance can be neglected. The variables K_T (armature constant) and K_b (motor constant) are proportionality constants. The ODE (49) can be written in matrix form as

$$\frac{dx(t)}{dt} = Ax(t) + au(t) \qquad (50)$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-K_T K_b}{RJ} \end{bmatrix}$, $a = \begin{bmatrix} 0 \\ \frac{K_T}{RJ} \end{bmatrix}$, $u(t) = V_{in}(t)$ and $x(t) := \begin{bmatrix} \theta(t) \\ \frac{d\theta}{dt} \end{bmatrix}$.

In this application a stochastic voltage source (white noise) and a stochastic resistance (colored noise) are considered. The noise in voltage is due to factors such as ripple voltage generated by incomplete suppression of the alternating waveform after rectification whereas the noise in the resistance is by discreteness of electric charge (thermal noise), see Rawat and Parthasarathy (2008); Kolarová and Brancík (2016); Kolarová (2005). Random effects due to both white and colored noise can be included by replacing the input and internal parameters in the deterministic model (50) through a random process as shown in the following.

SDE driven by a mixture noise. Consider a stochastic voltage source of the form $V_{in}(t) = V_{in}(t) + \sum_{i=1}^{n} \alpha_i dW_i(t)$, therefore, the SDE (50) is replaced by

$$dx(t) = (Ax(t) + au(t))dt + \frac{K_T}{RJ}\sum_{i=1}^{n} \alpha_i dW_i(t), \qquad (51)$$

with $\sum_{i=1}^{n} \alpha_i = 1$.

SDE driven by a white noise and a colored noise. In this case, it adds noise in both the voltage source (white noise) and the resistance (colored noise), i.e., voltage and the resistance are replaced by $V_{in} + \eta\xi_1(t)$, and $R+\beta w(t)$, respectively, where $\xi_1(t)$ is a white noise and $w(t)$ is the solution of the Ornstein-Uhlenbeck's differential equation (9). Under these two noises the stochastic differential equation that model the angular position and angular velocity in a DC motor is

$$dx(t) = (A(w(t))x(t) + a(w(t)))u(t)dt + b(w(t))dW(t) \qquad (52)$$

where $x(t) := \begin{bmatrix} w(t) \\ \theta(t) \\ \frac{d\theta}{dt} \end{bmatrix}$, $A(w(t)) = \begin{bmatrix} -\rho & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{-K_T K_b}{(R+\beta w(t))J} \end{bmatrix}$,

$a(w(t)) = \begin{bmatrix} 0 \\ 0 \\ \frac{K_T}{(R+\beta w(t))J} \end{bmatrix}$, $b(w(t)) = \begin{bmatrix} \sigma\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\eta K_T}{(R+\beta w(t))J} \end{bmatrix}$,

$W(t) = \begin{bmatrix} \overline{W}(t) \\ 0 \\ W_1(t) \end{bmatrix}$, and $u(t) = V_{in}(t)$.

Comparison of the solutions from (50), (51) and (52) with experimental data.

A motor with Encoder of 6V / 12 V (4000/8000 RPM) and 30 mA, was used to carry out the experiment. The electrical variables values of the motor are $R = 5.2\ \Omega$, $J = 0.01\ kg.m^2$, $K_T = 0.75\ N.mA^{-1}$, $K_b = 0.75\ V\left(Rad.s^{-1}\right)^{-1}$, and $V_{in} = 12\ V$. The experiment consists of connecting the direct current (DC) source to the DC motor and measuring the angular position $\theta(t)$ and calculating angular velocity $d\theta(t)/dt$. This process was repeated 20 times with a duration of $T = 0.5$ seconds.

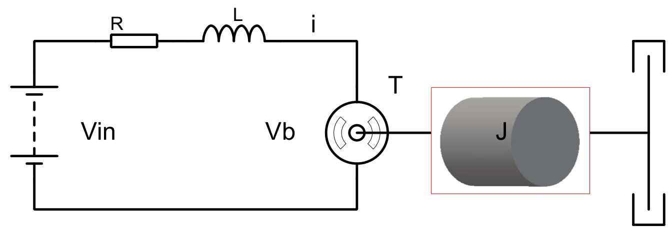

Fig. 1. DC motor.

Table 1. Comparison of the measurements of the angular position and angular velocity variables vs solution of ODE and SDE.

Comparison of Measurements vs	RMS Error θ (Rad)	RMS Error $d\theta/dt$ (Rad/s)
Solution of ODE	2.9035	2.1922
SDE driven by a white noise and a colored noise	2.49014	1.8605
SDE driven by a mixture noise	2.9039	2.2023

To obtain the measurements a Compact Rio 9068, a digital inputs module 9375, an encoder connected to the motor shaft and the Labview software were used.

Table 1 shows the root mean square error (RMSE) between the experimental data (measurements) and the analytical solutions of the ODE (50) and the SDEs (51) and (52). These RMSE show that the SDE (52) presents the best good fit to the experimental data (real system). Therefore, we solve the finite-horizon LQR optimal control problem considering that the dynamic system for the DC motor evolves as in (52).

Finite-horizon LQR optimal control problem.

In this application $U = [-12, 12]$. Fig. 2 shows the simulation of the angular position and angular velocity of the DC motor considering the dynamic system (52) and the optimal control $u(t)$ as in (32). Observe that both the angular position and angular velocity stabilize in zero at $t = 2.5$ s. The values of parameters used in the simulation are $\sigma = 0.1, \eta = 1, \rho = 1.0$, $\beta = 1.0$, the matrix $R = 100000$, Q is the identity matrix, and the initial states are $\theta(0) = 10$ *rad* and $\frac{d\theta(0)}{dt} = 10$ *rad* s^{-1}. The Figs. 2 and 3 show that both the states $\theta(t)$, $d\theta(t)/dt$ and the value function $J^*(t,x)$ tends to zero since in this application the objective of the LQR controller is stabilize the dynamic system to zero, precisely.

6.2 *Single-phase grid-connected inverter with LCL filter*

The mathematical model for the grid-connected pulse width modulated inverter was developed in Tang et al. (2012). The noises present in the real system did not into account. However, to illustrate our theory, we consider that the equivalent series resistors (R_1 and R_2) are influenced by a colored noise (thermal noise) whereas the inverter output voltage v_o is affected by a white Gaussian noise, ($v_o(t) = v_o(t) + \sigma_1 \xi(t)$). So, the model studied is

$$dx(t) = (A(w(t))x(t) + B_1 v_o(t) + B_2 v_s(t))dt + \sigma B_1^T dW(t) \tag{53}$$

with $x(t) := \begin{bmatrix} i_1(t) \\ i_2(t) \\ v_c \end{bmatrix}$,

$$A = \begin{bmatrix} -R_1 + w(t)/L_1 & 0 & 1/L_1 \\ 0 & -R_2 + w(t)/L_2 & 1/L_2 \\ 1/C & -1/C & 0 \end{bmatrix},$$

$$B_1 := \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix},\ B_2 := \begin{bmatrix} 0 \\ -1/L_2 \\ 0 \end{bmatrix},\ \sigma = \begin{bmatrix} \sigma_1 \\ \sigma_1 \\ \sigma_1 \end{bmatrix},$$

where L_1 is inverter side inductance; L_2 is grid side inductance; R_1 is equivalent series resistor of L_1; R_2 equivalent series resistor of L_2; i_1 inverter output current; i_2 grid side current; v_c capacitor voltage; v_s is the grid side voltage. The main function of the inverter consists with to convert the direct current (DC) power delivered by photovoltaic panels to alternating current (AC) power. In order to transmit the AC power, the inverter must follow voltage and current reference signals, therefore, a LQR controller can be applied. In this case, the objective of the LQR controller consists of achieving maximum power delivery from the inverter to the grid.

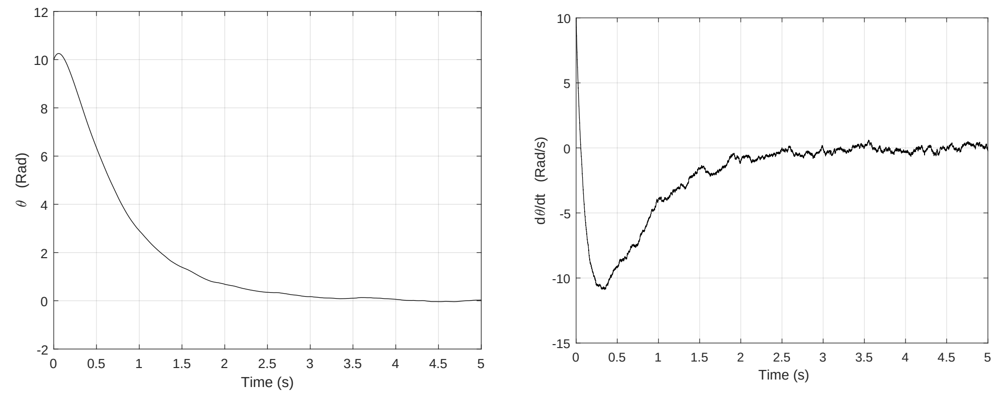

Fig. 2. Angular position θ (left) and angular velocity $d\theta/dt$ (right) for the finite-horizon LQR problem.

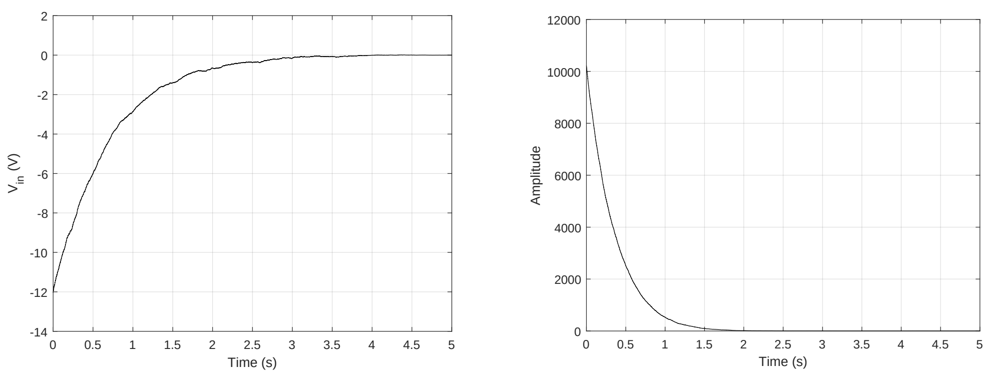

Fig. 3. Optimal control $f^*(t,x)$ (left) and value function $J^*(t,x)$ (right) for the finite-horizon LQR problem.

The reference signals proposed in Tang et al. (2012) are: $i_{2ref} = 20\sin(\omega t)$, $v_{sref} = 311\sin(\omega t)$,

$$i_{1ref} = C\frac{dv_s}{dt} + i_{2ref}, \text{ and } v_{cref} = L_2\frac{di_{2ref}}{dt} + v_s.$$

LQR optimal control.

A LQR controller can be found linearizing $A(w(t))x(t) + B_1 v_o(t) + B_2 v_s(t)$ around an operating point $(x_{ref}, v_{cref}, v_{sref})$ assuming that the deviation $v_s - v_{sref} = 0$.

In the infinite horizon case, the LQR optimal control is (40) with $x(t)-x_{ref}$ in lieu of $x(t)$ and $a(w) \equiv B_1$. The value function is (41) with K satisfying the Riccatti equation (42) and $g(w)$ is the solution of the ODE (43). In the simulation of (9), (32), (37) and (53) the parameters used were; $L_1 = L_2 = 1mH$, $R_1 = R_2 = 0.01\Omega$, $C = 20\mu F$, DC bus voltage V_{dc} equals $400V$, switching frequency of $10kHz$, grid voltage (line to line) of $380V$, and grid frequency of 50 Hz, $\sigma = 1$, $\rho = 2$, $\alpha = 1$. A random behavior in the states i_1 and i_2 is observed because the resistances were modeled with colored noise, see Fig. 4. It can be seen that these states follow the reference signal very precisely, so the stochastic LQR controller has good steady-state performance and can be more nearby to the real system. The law control $f^*(x)$ and the value function $J^*(x)$ are displayed in Fig. 5, note that $f^*(x)$ is in interval $[-60, 60]$ while that $J^* \in [0, 6\times 10^4]$.

6.3 *Temperature control of a non-isothermal continuous stirred-tank reactor (CSTR)*

In the work Meghna et al. (2017), the authors used a LQR controller in order to maintain the temperature of the reactant mixture to the given set point by means of coolant medium. It's well known that in steady state, the heat removed by the coolant medium should be equal to the heat that is produced by the reaction. Therefore, its objective was to control the temperature of the product manipulating the temperature of the coolant.

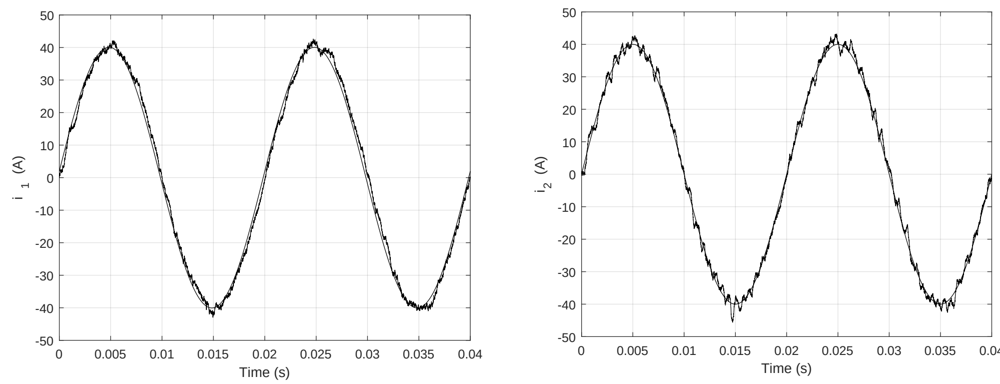

Fig. 4. Infinite horizon LQR for single phase grid connected inverter.

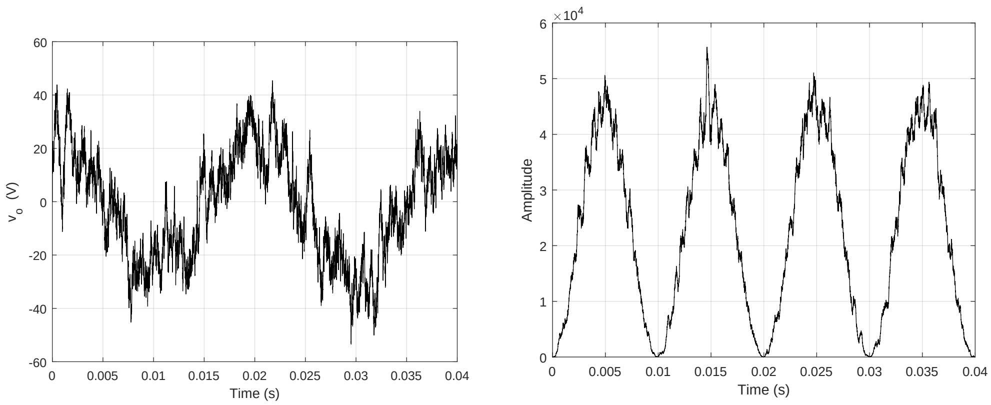

Fig. 5. Optimal control f^* (left) and value function J^* (right) for infinite-horizon LQR problem.

The concentration C_a and temperature T in the non-isothermal CSTR are modelled by the nonlinear stochastic differential equations

$$\frac{dC_a}{dt} = \frac{Q}{A}(C_{ai} - C_a) - K_o e^{-\frac{E}{RT}} C_a, \tag{54}$$

$$\frac{dT}{dt} = \frac{Q}{T}(T_i - T) + J K_o e^{-\frac{E}{RT}} C_a - \frac{UA}{\rho c_p V}(T_c - T). \tag{55}$$

The parameters in (54)-(55) are: C_a := concentration of A in the CSTR (mol/m^3); Q := volumetric flow rate (m^3/s); V := volume of the CSTR (m^3); C_{ai} := inlet concentration (mol/m^3); K_o := constant ($1/sec$); E := activation energy (J/mol); R := universal gas constant ($J/molk$); J := heat of reaction (J/mol); U := overall heat transfer coefficient ($W/m^2 - K$); A := Area (m^2); T_i := inlet temperature (K); T := temperature in the CSTR (K); T_c := coolant temperature (K); ρ := density of the A-B mixture (Kg/m^3).

Now, we linearized (54)-(55) about the steady state $(C_{ass}, T_{ss}, T_{css})$ by means of Taylor series neglecting the higher order terms. To this end, we consider

$$f(C_a, T, T_c) := \frac{Q}{A}(C_{ai} - C_a) - K_o e^{-\frac{E}{RT}} C_a,$$

$$g(C_a, T, T_c) := \frac{Q}{T}(T_i - T) + J K_o e^{-\frac{E}{RT}} C_a - \frac{UA}{\rho c_p V}(T_c - T),$$

and

$$\frac{dC_a}{dt} - f(C_{ass}, T_{ss}, T_{css}) = \frac{\partial f}{\partial C_a}(C_a - C_{ass}) + \frac{\partial f}{\partial T}(T - T_{ss}) + \frac{\partial f}{\partial T_c}(T_c - T_{css}), \tag{56}$$

$$\frac{dT}{dt} - g(C_{ass}, T_{ss}, T_{css}) = \frac{\partial g}{\partial C_a}(C_a - C_{ass}) + \frac{\partial g}{\partial T}(T - T_{ss}) + \frac{\partial g}{\partial T_c}(T_c - T_{css}). \tag{57}$$

In the steady state $\frac{dC_a}{dt} = \frac{dT}{dt} = 0$ implying that $f(C_{ass}, T_{ss}, T_{css}) = g(C_{ass}, T_{ss}, T_{css}) = 0$. Taking $C_a = 4$, $Q = 1$, $V = 100$, $C_{ai} = 1$, $K_o = 0.01$, $E = 8697$, $R = 1$, $A = 1$, $J = 10^4$, $U = 10^4$,$T_i = 289$, $\rho = 100$, $c_p = 100$, $T_{ss} = 365$, $T_{css} = 289$ and $C_{ass} = 4$, we get

$$\frac{\partial}{\partial C_a} f(C_a,T,T_c) = -\frac{Q}{A} - K_o e^{-\frac{E}{RT}}|_{T=T_{ss}} = -0.01, \quad (58)$$

$$\frac{d}{dT} f(C_a,T,T_c) = -\frac{C_a E K_o e^{-\frac{E}{RT}}}{RT^2}|_{T=Tss,C_a=C_{ass}} = 0, \quad (59)$$

$$\frac{d}{dT_C} f(C_a,T,T_c) = 0, \quad (60)$$

$$\frac{\partial}{\partial C_a} g(C_a,T,T_c) = J K_o e^{-\frac{E}{RT}}|_{T=T_{ss}} = 0, \quad (61)$$

$$\frac{\partial}{\partial T} g(C_a,T,T_c) =$$

$$-\frac{AU}{Vc_p\rho} - \frac{QT_i}{T^2} + \frac{EJK_oC_a e^{-\frac{E}{RT}}}{RT^2}|_{T=Tss,C_a=C_{ass}} = 0.006, \quad (62)$$

and

$$\frac{\partial}{\partial T_c} g(C_a,T,T_c) = -\frac{AU}{Vc_p\rho} = -0.01. \quad (63)$$

Substituting (58)-(63) in (56) and (57) we obtain the linearized mathematical model for the concentration and temperature in the non-isothermal CSTR

$$\begin{aligned} \frac{dC_a}{dt} &= -0.1(C_a - C_{ass}), \\ \frac{dT}{dt} &= 0.006(T - T_{ss}) + 0.01(T_c - T_{css}). \end{aligned}$$

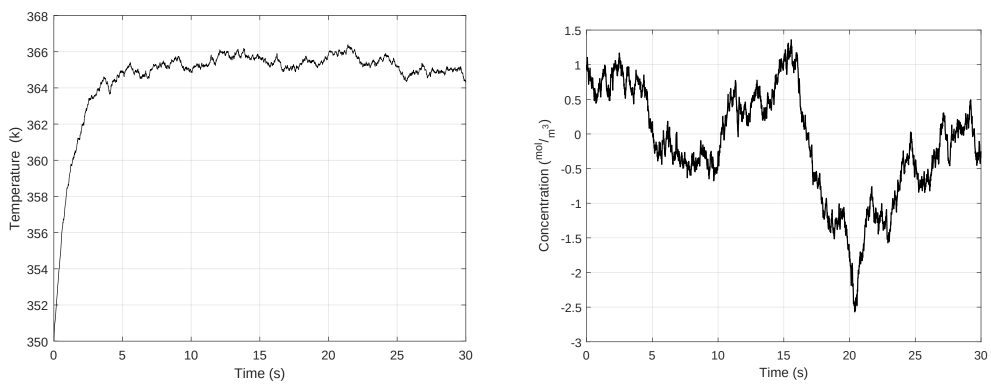

Fig. 6. Temperature (left) and concentration (right) of the reactant mixture in the CSTR.

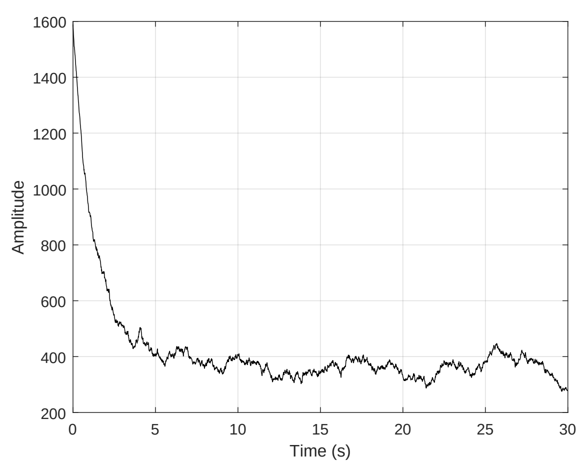

Fig. 7. Controller: temperature of the coolant.

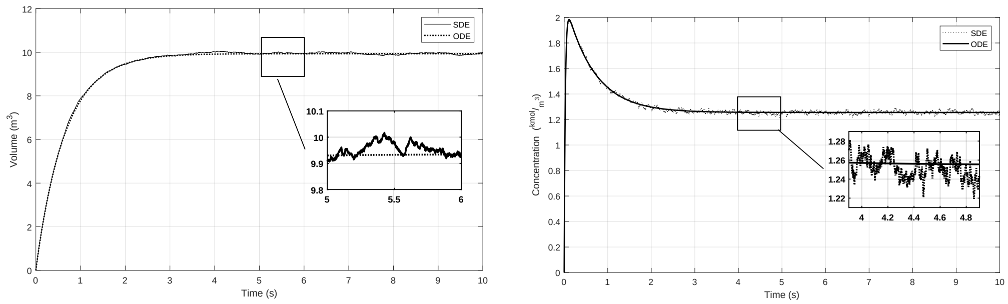

Fig. 8. Volume and concentration of liquid in the CSTR.

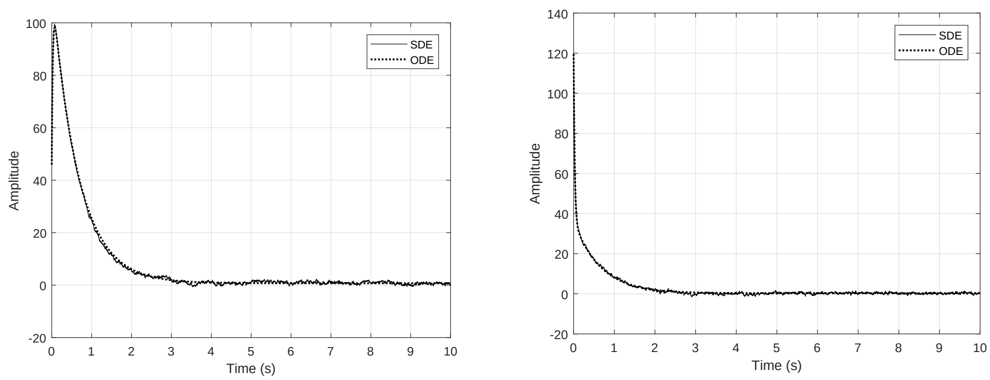

Fig. 9. Input flows $(u_1(t), u_2(t))$.

Finally, aiming at designing the finite-horizon stochastic LQR controller (17)-(32)-(48), we define $C_{al}(t) := C_a(t) - C_{ass}; T_l(t) := T(t) - T_{ss}$ and $u(t) := T_c(t) - T_{css}$ to get the linear SDE driven by a mixture noise

$$dx(t) = \begin{bmatrix} -0.01 & 0 \\ 0 & 0.006 \end{bmatrix} \begin{bmatrix} C_{al}(t) \\ T_l(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \begin{bmatrix} 0 \\ u(t) \end{bmatrix} + \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} + \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} dW_3(t) \\ dW_4(t) \end{bmatrix}, \tag{64}$$

where $x(t) := \begin{bmatrix} C_{al}(t) \\ T_l(t) \end{bmatrix}$, $W_i(\cdot)$ and $W_j(\cdot)$ are independent Brownian motions for $i \neq j$, $i, j = 1,2,3,4$, with amplitudes α_1 and α_2, respectively. In the simulation $\alpha_1 = 0.5, \alpha_2 = -0.5$, $R = 1$ and

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 2500 \end{bmatrix}. \tag{65}$$

Figs. 6 and 7 show that the temperature of the reactant mixture achieved given set point (level) $T_{ss} = 365, T_{css} = 289$. Therefore, we can conclude that the stochastic LQR is a good control that capture external noises possibly caused by the operation of the pump.

6.4 *Volume and concentration control of an CSTR*

In order to maintain the concentration and volume of liquid in the CSTR to the given set point (level) a LQR controller is designed. The finite-horizon stochastic LQR controller (17)-(32)-(48) based on the linearized mathematical model of the CSRT is proposed in this section. We recommend the reader to see Bin Poyen et al. (2013) for more details.

In the CSTR there are two time varying inlets to the tank with flow rates $F_1(t)$ and $F_2(t)$. The dissolved material concentrations of both the inlets are c_1 and c_2, respectively, with $c_1 \neq c_2$. The outgoing flow has a flow rate $F(t)$. It is assumed that the tank is continuously stirred and mixed well, so that the concentration of the outlet equals the concentration in the tank i.e., $c(t)$. In the steady-state situation, all quantities are assumed to be constant, say V_0 for the volume, c_0 for the concentration and F_{10}, F_{20} for the flow rates. Let $V(t)$ be the volume of the fluid in the tank and define $\eta(t) := V(t) - V_0$, $\beta(t) := c(t) - c_0$, $u_1(t) := F_1(t) - F_{10}$ and $u_2(t) := F_2(t) - F_{20}$. The volume and the concentration in the tank are modelled by the stochastic differential equation with mixture noise

$$dx(t) = \begin{bmatrix} -0.01 & 0 \\ 0 & -0.02 \end{bmatrix} \begin{bmatrix} \eta(t) \\ \beta(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -0.25 & 0.75 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} + \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}, \quad (66)$$

where $x(t) := \begin{bmatrix} \eta(t) \\ \beta(t) \end{bmatrix}$, $W_i(t)$ and $W_j(t)$ are independent Brownian motions for $i \neq j$ $i, j = 1, 2, 3, 4$, and $\alpha_1 + \alpha_2 = 0$. The added external noise in (66) may be due to the operation of pumps, affecting flow rates. The set point in our simulation is $V_0 = 10\ Kmol/m^3$, $c_0 = 1.25\ Kmol/m^3$ $F_{10} = 0.015\ m^3/sec$, $F_{20} = 0.005$, m^3/sec, $\alpha_1 = 0.8, \alpha_2 = 0.1$ and $T = 10\ sec$. Figs. 8 and 9 show that the volume, the concentration, as well as, the flow rates of the liquid in the tank achieved given set point. Therefore, the stochastic LQR controller with mixture noise, maintain desired volume and concentration of liquid in the CSTR and moreover, capture possible external noises.

Concluding remarks

This paper concerns with LQR- stochastic optimal control problems where the dynamic systems are affected by a combination of white and colored noises. Our main results can be summarized as follows,

1. It shows the general procedure to calculate through Itô's lemma the infinitesimal generator of the Markovian diffusion processes. This is a crucial step to raise the HJB-equations in the dynamic programming approach to solve stochastic optimization problems.
2. The infinitesimal generators for the controlled linear SDE studied were explicitly given.
3. The LQR stochastic optimal control problems where the system's dynamic evolved according to: (a) a multiplicative and additive white noise, (b) a mixture white noise and (c) a white and colored noise, were explicitly resolved, see Propositions 5.1 and 5.2 and Remark 5.4.
4. The value functions and optimal policies, as well as, algebraic and differential Riccati equations were analytically found.
5. The operator in (22) is the infinitesimal generator of a diffusion process of the form:

   $$dx(t) = b(t, x(t), w(t), u(t))dt + \sigma(x(t))dW(t),$$

   where $w(t)$ evolves as in (15), i.e., $w(t)$ is not necessarily a colored noise. Therefore, we obtained the HJB-equation associated to more general optimal control problems which allow us to study other applications. In fact, our results can be extended to LQR-OCPs where the system's dynamic is a hybrid dffusion process, Pola et al. (2003), Blom (2003), Mao et al. (2007).

Base on our experience the type of noise impacting the dynamic system and the way it will be added has to be check before to solve the optimization problem. In our case, the LQR-optimal control problem associated with the DC motor dynamic which evolves with a combination of white and colored noise was used because is the best fit to experimental data.

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