Revista Mexicana de Ingeniería Química

Vol. 11, No. 2 (2012) 269-278

RISING VELOCITY FOR SINGLE BUBBLES IN PURE LIQUIDS

VELOCIDAD DE ASCENSO DE BURBUJAS AISLADAS EN LÍQUIDOS PUROS

S. Baz-Rodríguez[1]*, A. Aguilar-Corona[2] and A. Soria[1]

[1]*Departamento de Ingeniería de Procesos e Hidráulica, Universidad Autónoma Metropolitana - Iztapalapa, San Rafael Atlixco 186, Col. Vicentina, México, D. F. CP. 09340, México.*
[2]*Facultad de Ingeniería Mecánica, Universidad Michoacana de San Nicolás de Hidalgo, Francisco J. Mujica s/n, Ciudad Universitaria, CP. 58040, Morelia, Michoacán, México.*

Received October 9, 2011; Accepted March 10, 2012.

Abstract
An equation to predict the terminal rise velocity of single bubbles in stagnant Newtonian liquids is proposed. The formulation combines a force balance obtained from the boundary layer theory for non-distorted bubbles and an analytic equation coming from a mechanic energy balance. Without including geometric parameters, which are difficult to assess, it is assumed that the weighting of dominant forces is enough to adequately predict the terminal velocity in both the intermediate and inertial motion regimes. The proposed equation shows good agreement with experimental data from bubbles rising in pure liquids. Moreover, for bubbles rising in clean water, the effect of helical trajectories was estimated from experimental data trends and included in the formulation as a correction factor for the terminal velocity.

Keywords: single bubble motion, terminal velocity equation, oscillatory bubble path, pure liquids.

Resumen
Se propuso una ecuación para predecir la velocidad terminal del ascenso de burbujas aisladas en líquidos Newtonianos. La formulación combina un balance de fuerzas obtenido de la teoría de capa límite para burbujas esféricas con una ecuación analítica proveniente de un balance de energía mecánica. Sin la inclusión de parámetros geométricos que son difíciles de determinar, se asume que la ponderación de las fuerzas dominantes es suficiente para predecir adecuadamente la velocidad terminal en los regímenes de movimiento intermedio e inercial. Las predicciones de la ecuación propuesta presentan buen ajuste con datos experimentales para burbujas ascendiendo en líquidos puros. Además, para burbujas ascendiendo en agua pura, se estimó el efecto de las trayectorias helicoidales a partir de la tendencia de datos experimentales y se incluyó en la formulación como un factor de corrección de la velocidad terminal.

Palabras clave: movimiento de burbujas aisladas, ecuación de velocidad terminal, trayectoria oscilatoria de burbujas, líquidos puros.

1 Introduction

Motion analysis of a single bubble rising in quiescent liquid is useful to understand and describe gas-liquid bubbly flows. Let us consider a train of bubbles rising unconfined through still liquid. Its rise velocity can be approximated by that of a single bubble plus the velocity defect caused by the wakes (Marks, 1973). If one is dealing with bubble columns, the slip velocity between phases can be expressed in terms of a single bubble terminal velocity and the gas volume fraction (Shah *et al*., 1982; Ruzicka *et al*., 2001). These facts emphasize the importance of having equations that predict the terminal rise velocity of single bubbles, sin-

*Corresponding author. E-mail: bazrodriguez@gmail.com
Tel.: +52 55 58044648 Ext. 237

ce it is a fundamental parameter in gas-liquid flows.

The shape and terminal velocity of isolated bubbles vary as the equivalent diameter increases, according to three regimes (Tomiyama *et al*., 2002): 1) dominant viscosity, 2) intermediate region, where viscous, surface tension and inertial effects should be taken into account, and 3) dominant inertia. Out of the three regimes, the intermediate one presents major difficulties since the force interactions and their effect on the terminal velocity is not yet well understood. Moreover, the terminal velocity is particularly sensitive to the presence of contaminants and even to the initial deformation induced by injection (Tomiyama *et al*., 2002; Celata *et al*., 2007).

The intermediate regime spans an equivalent diameter range from 0.07 to 0.65 cm for air bubbles ascending through pure water (Mendelson, 1967; Loth, 2010). This corresponds to particle Reynolds numbers (Re) between 80 and 1500 for normal pressure and temperature conditions, approximately. This working interval is usual in many industrial systems. When Re > 200, bubble shapes are oblate ellipsoidal with fore-aft symmetry. This symmetry gradually brakes as Re increases, until the bubble motion attains the inertial spherical cap regime. Besides, for Re ~ 650-685, the onset of path instability occurs, and a transition from rectilinear to zig-zag or helical trajectories is observed (Duineveld, 1995; Sanada *et al*., 2008; Veldhuis *et al*., 2008).

There exist three basic approaches to establish the terminal velocity (Kulkarni and Joshi, 2005):

- Force balance. The terminal velocity equations result from a force balance between drag and buoyancy. The most representative solutions are obtained from drag in creeping and potential flows. This approximation can be adequate to predict the rise velocity of small bubbles when viscosity effects are still dominant.

- Dimensional analysis. Dimensionless groups are determined from the leading variables that govern the bubble motion. A functional relation is proposed for these groups, and the adjustable parameters are fitted from experimental or numerical data (Rodrigue, 2001).

- Wave analogy. Interfacial disturbances are assumed for bubbles whose dynamics is similar to propagating waves in an ideal fluid, and then the terminal velocity is estimated as a function of the equivalent diameter and the fluid properties (Mendelson, 1967). Even if some doubt has been cast on the original approach because of its lack of physical basis (Lehrer, 1976; Fan and Tsuchiya, 1990; Bozzano and Dente, 2001), the wave analogy approach can span some later investigations, which obtained equations with the same basic form (Lehrer, 1976; Zudin, 1995). Predictions using this approach fit the main trend after the bubble size corresponding to the terminal velocity local maximum is attained (0.14-0.18 cm for air in water, see Fig. 1).

Other formulations for bubble terminal velocities have been proposed from some detailed bubble shape and external flow considerations (Moore, 1965; Bozzano and Dente, 2001; Tomiyama *et al*., 2002). However, these approaches often require additional data such as aspect ratio (between minor and major axes) and (hemispherical) bubble distortion factor. These parameters are difficult to obtain. Some investigations report the aspect ratio from experimental results on pure and contaminated liquids as a function of the Eötvos number (Eo) (Wellek *et al*., 1966; Okawa *et al*., 2003; Sanada *et al*., 2008). Sometimes the Weber number (We) is used instead (Moore, 1965; Wellek *et al*., 1966), which seems more appropiate to establish functional relations with this geometrical parameter (Celata *et al*., 2007). Nevertheless, expressing the aspect ratio in terms of We involves its own dependency on the terminal velocity. Besides, for pure liquids it is difficult to estimate aspect ratios or distortion factors even experimentally (Tomiyama *et al*., 2002), in contrast with the case of liquids contaminated with surfactants, which act by damping bubble shape oscillations (Celata *et al*., 2007).

The inclusion of drag coefficients with non-linear Re dependency, or aspect ratio that depends on We in any terminal velocity formulation implies solving non-linear equations, often with fractional exponents. This is inconvenient when the terminal velocity is not the main objective of an engineering calculation. For this reason, a simple equation to predict the isolated bubble terminal velocity is proposed in the present work. If geometrical considerations are avoided, we believe that the proper weighting of the dominant forces is enough to assess this important parameter for the intermediate and inertial regimes.

The terminal velocities reported in literature correspond to the axial component of the bubble center of mass velocity vector. However, for the case of air rising through water, it is important to take into account that rectilinear trajectories become unstable above some critical *Re*. For this reason, and considering a helical trajectory, a correction factor was obtained as a function of the mean slope of the helix. This procedure improves the fit to experimental data in the equivalent diameter range that correspond to the oscillatory motion of the bubble.

2 Terminal velocity formulation

Considering the intermediate region, inertial effects are combined with viscosity and surface tension effects. This suggests that an equation for the bubble terminal velocity, which is valid when viscous effects are still important, can be combined with another one that is valid when surface tension effects are significant (Fan and Tsuchiya, 1990; Jamialahmadi *et al*., 1994). In the former case it is reasonable to propose an equation developed from the force balance approach. In the later it would be sensible to use a wave analogy approach. Proceeding this way, an equation can be obtained, which is valid for a wide range of equivalent bubble diameters.

The proposed formulation for the terminal velocity is given here by the following combination of the corresponding effects.

$$V_T = \frac{1}{\sqrt{\frac{1}{V_{T1}^2} + \frac{1}{V_{T2}^2}}} \tag{1}$$

where V_{T1} is the rise velocity when viscous effects are still important and V_{T2} is the corresponding velocity when surface tension effects are significant. The form of Eq. (1) was suggested by Jamialahmadi *et al*. (1994) and remains in use (Kulkarni and Joshi, 2005; Cai *et al*., 2010; Duangsuwan *et al*., 2011); also, a similar form but with fitting parameters was previously proposed by Fan and Tsuchiya (1990). When viscous effects are important and the bubble diameter is small, Eq. (1) ensures that the contribution of V_{T2} is neglected. This is important because the terminal velocity equations from wave analogy yield values too high below the intermediate velocity maximum, and they are preferably valid until significant surface tension effects appear. The opposite occurs with V_{T1} at larger bubble diameter. In the intermediate region, Eq. (1) is just the terminal rising velocity coupled between the two velocities V_{T1} and V_{T2}.

Jamialahmadi *et al*. (1994) proposed the equation developed by Hadamard (1911) and that of Mendelson (1967) for V_{T1} and V_{T2}, respectively. Nevertheless, Hadamard equation is only valid for Reynolds numbers smaller than unity (Tomiyama *et al*., 2002) and Mendelson equation seems not to have an adequate basis to justify its deduction (Lehrer, 1976; Bozzano and Dente, 2001).

When viscous effects are losing their influence, the terminal velocity for a bubble in potential flow can be used (Levich, 1962)

$$V_{Tpot} = \frac{1}{36}\frac{\Delta\rho g d_e^2}{\mu_L} \tag{2}$$

where $\Delta\rho$ is the density difference between the liquid and gaseous phases, d_e is the diameter of a spherical bubble of equivalent volume, μ_L is the dynamic viscosity of the liquid and g is the acceleration of gravity. Contrary to Hadamard solution, which overestimates the terminal velocity for moderate Re, the potential solution underestimates them. Using the governing equations of motion for the boundary layer on a spherical bubble, Moore (1963) obtained a correction for the potential solution of drag force which fits particularly well for moderate regime motion ($50 < Re < 200$), before the surface tension and inertial effects start to cause deviations from the spherical bubble shape. Consequently, the boundary layer solution must be more appropriate when the bubble rise at moderate *Re*.

Buoyancy and drag as expressed by Moore (1963) gives the following equilibrium:

$$6\pi\mu_L d_e V_{T1}\left(1 - \frac{2.21}{Re_1^{1/2}}\right) = \frac{1}{6}\pi d_e^3 \Delta\rho g \tag{3}$$

where

$$Re_1 = \frac{d_e V_{T1}\rho_L}{\mu_L} \tag{4}$$

Rearranging Eq. (3), the solution for the terminal velocity for potential flow appears

$$\frac{1}{V_{Tpot}^{1/2}(gd_e)^{1/2}} V_{T1}^{3/2} - \frac{V_{Tpot}^{1/2}}{(gd_e)^{1/2}} V_{T1}^{1/2} - 0.36833 = 0 \tag{5}$$

Expanding a Taylor series around the potential flow solution, and truncating to the second order term, an equation for the terminal velocity is obtained. The above allows obtaining an approximate solution for the bubble velocity avoiding numerical solving of Eq. (5). The resulting equation is as follows

$$V_{T1} = V_{Tpot}\left[1 + 0.73667\frac{(gd_e)^{1/2}}{V_{Tpot}}\right]^{1/2} \tag{6}$$

Eq. (6) is valid for spherical isolated bubbles rising under the effect of a hydrodynamic boundary layer and buoyancy force.

When surface tension effects are important, Lehrer (1976) states a mechanical energy balance generated by bubble displacement in the liquid:

$$\frac{1}{6}\pi d_e^3 \frac{1}{2}\rho_L v^2 = \sigma \pi d_e^2 + \frac{1}{6}\pi d_e^3 \Delta\rho g d_e \qquad (7)$$

where σ is the surface tension and v is the liquid velocity when it is displaced by the bubble passing by.

The right hand side of Eq. (7) is the potential energy increase of the displaced liquid due to surface tension and buoyancy during the bubble motion over a distance d_e. This potential energy is then transformed in kinetic energy [left hand side of Eq. (7)] as the bubble moves, and finally it is dissipated by the wake. The liquid motion is considered uniformly accelerated. Assuming that the necessary time for the potential energy to be transformed into kinetic energy is that corresponding to a displacement d_e of the bubble moving at a steady velocity, the following equation is obtained (Lehrer, 1976).

$$V_{T2} = \left(\frac{3\sigma}{\rho_L d_e} + \frac{g d_e \Delta\rho}{2\rho_L} \right)^{1/2} \qquad (8)$$

Eq. (8) can predict the terminal velocity when the dominant effects come from surface tension and inertia. According to the combination proposed by Eq. (1) it is possible to calculate the terminal velocity for a large range of bubble sizes in pure liquids for the intermediate and inertial regimes.

3 Results and discussion

Predictions from Eqs. (6) and (8), combined according to Eq. (1), were compared with experimental data from Haberman and Morton (1953), Peebles and Garber (1953), Okazaki (1964), Aybers and Tapucu (1969), Duineveld (1995), Blandín-Arrieta (1997), Leifer *et al*. (2000), Okawa *et al*. (2003), Talaia (2007) and Sanada *et al*. (2008) for gas bubbles ascending through various pure liquids, including distilled and ultra-purified water.

Fig. 1 shows the comparison of the experimental data for distilled water case in a wide equivalent diameter range with the terminal velocity predicted from equations. The proposed formulation in this work properly predicts the typical behavior of the terminal velocity along the whole range. Predictions of three other formulations are also presented. Two of them propose a weighting similar to that of Eq. (1) (Fan and Tsuchiya, 1990; Jamialahmadi *et al*., 1994), and the other one was obtained by dimensional analysis (Rodrigue, 2001).

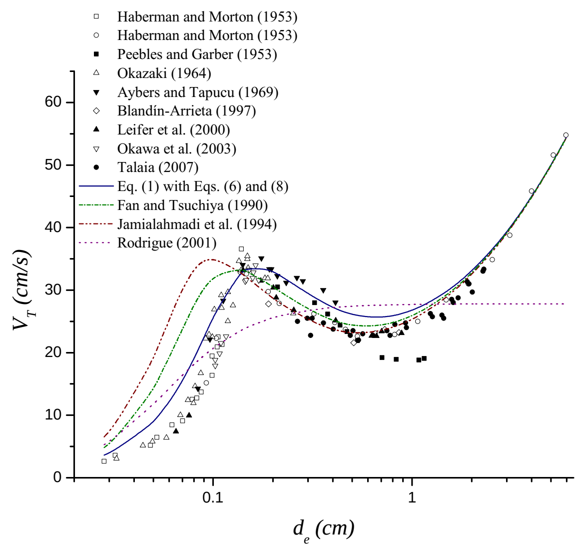

Fig. 1. Comparison between predictions of equations and experimental data for bubble terminal velocity through distilled water ($7.4 < Re < 32500; 0.002 < We < 245; 0.01 < Eo < 475$) at 20-28 °C. Markers correspond to experimental data, and lines to predictions from the equations.

The prediction of terminal velocity by Jamialahmadi *et al*. (1994) equation fails before the local maximum, and does not fit the experimental data at this maximum. This happens because the solution proposed by Hadamard -and used by Jamialahmadi *et al*. (1994) for dominant viscous effects-, overestimates the terminal velocity value for Re larger than one. A similar situation occurs with Fan and Tsuchiya (1990) equation, which is based on a parameter fit. Additionally, Rodrigue (2001) equation results particularly inadequate to predict the existence of local extrema on the typical terminal velocity curve.

Distilled water can still contain contaminants that affect the rising motion of a single bubble. The presence of such pollutants, especially surfactants, changes the interface characteristics, rigidifying it and reducing its shape oscillations during rise. Duineveld (1995) and Sanada *et al*. (2008) determined experimentally the rise velocity of bubbles moving through ultra-purified water with rectilinear trajectories ($20 < Re < 685$). Fig. 2 shows the comparison of those experimental data with predictions of the proposed formulation and some other from the literature, as the equation developed by Tomiyama *et al*. (2002). The latter equation requires the knowledge of the aspect ratio. For small equivalent diameters it is still acceptable to relate this parameter to the E_o (Celata *et al.*, 2007). Here, Sanada *et al.*

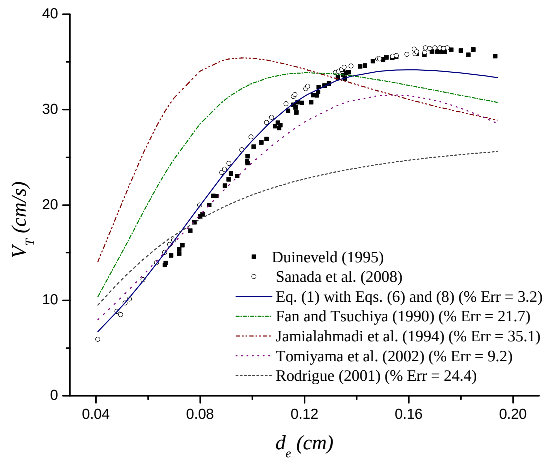

Fig. 2. Terminal velocity from experiments on ultra-purified water ($25 < Re < 685; 0.02 < We < 3.36; 0.02 < Eo < 0.5$) at 20-23 °C. Markers correspond to experimental data, and lines to predictions from the equations.

(2007) equation ($E_o < 0.5$) was used

$$E = \frac{1}{1 + 6.5Eo^{1.925}} \tag{9}$$

In Fig. 2 one can observe that the relative average error (percentage, % Err in the figures) is significantly smaller for the formulation proposed in this work (3.2 % absolute). Tomiyama *et al.* (2002) equation, which is the next better fit, obtain a relative average error of 9.2 %. This seems to confirm that viscous effects are concentrated in a thin boundary layer for the bubble size ranging from 0.04 to 0.18 cm of equivalent diameter, according to the approximation from Eq. (6).

As the bubble diameter increases, up to Eo about 0.5, the onset of path instability occurs (Duineveld, 1995; Sanada *et al.*, 2008; Velduis *et al.*, 2008), and trajectories become zig-zagging or helical. The corresponding bubble size range spans between the local extrema of the typical terminal velocity behavior (see Fig. 1). For pure water these extrema are between $d_e \sim 0.15 - 0.65$ cm, approximately (Mendelson, 1967; Loth, 2010), and according to the proposed formulation between $d_e \sim 0.17 - 0.67$ cm. These local extrema are important because they indicate significant transitions in the bubble rise mechanics. The local maximum coincides with the path instability onset, while the local minimum indicates the point where inertial effects become dominant over surface tension effects in Eq. (8).

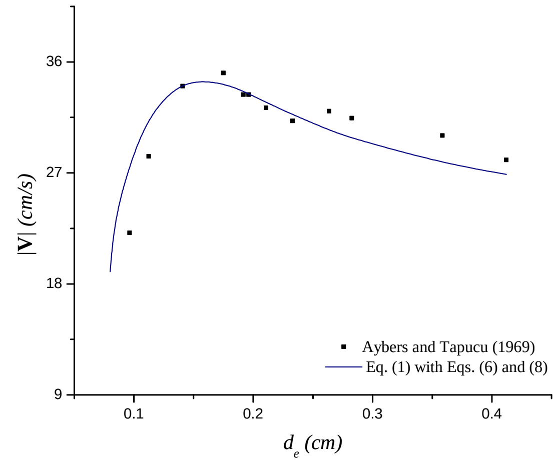

Fig. 3. Comparison between predictions of Eq. (1) and the single bubble velocity vector magnitude from Aybers and Tapucu (1969).

For larger bubble size, the rise regime is dominated by inertia, and bubbles adopt the characteristic spherical cap shape, with the trajectory becoming approximately rectilinear again.

For the terminal velocity between the local extrema, three bubble diameters are possible. The lowest diameter corresponds to a motion with rectilinear path, the middle diameter is embedded in the intermediate regime with oscillatory trajectory, and the largest diameter corresponds to a preferably inertial regime and spherical cap shape for the bubble. In this sense, the description of the bubble motion within the oscillating trajectory range is a very complex task, and it remains a research topic in recents numerical and experimental works (de Vries *et al.*, 2002; Mougin and Magnaudet, 2006; Shew and Pinton, 2006a; Veldhuis *et al.*, 2008).

Even if the predictions proposed in this work fit the experimental data with pure water for many practical applications (*Re* between 25 and 685) best than other predictions, they overestimate slightly the terminal velocity for larger bubbles. This occurs until the inertial regime is reached, as shown in Fig. 1. This overestimation might be caused by the presence of contaminants in the water or by oscillatory trajectories.

Aybers and Tapucu (1969) measured both the velocity magnitude and its axial component (terminal velocity) for bubbles in the equivalent diameter range between 0.1-1 cm following helical paths. Fig. 3 shows a comparison between the predictions of Eq. (1) [with substitutions of Eqs. (6) and (8)] to the obtained data for the velocity magnitude. The percent relative error has an average of 5.5 %, as compared with experimental data by Aybers and Tapucu (1969). This allows us to state that if the approximation fits into Eq. (1) (with $V_T \approx |V|$), then its axial component can be determined from the bubble trajectory. For the case of small bubbles ($d_e < 0.17$) and large ones ($d_e > 0.67$) the velocity module corresponds approximately to its axial component, or terminal, velocity. Yet it is necessary to know the trajectory to determine the velocity magnitude and its axial component for oscillating motion.

Helical trajectories are much more common that zig-zagging ones (Shew and Pinton, 2006a), and correspond to the observed motion well above the nozzle (Aybers and Tapucu, 1969). Zig-zagging trajectories are associated to an unstable wake, so these trajectories usually become helical at large (Ellingsen and Risso, 2001; Shew and Pinton, 2006a). As a result, it is advisable to assume that helical trajectories characterize the whole range of oscillatory motion, but with a stationary terminal velocity during the bubble ascent (Shew and Pinton, 2006a; Shew *et al.*, 2006).

From the vectorial description of a helical path, Aybers and Tapucu (1969) obtained the following relationship between the axial velocity component and its vectorial magnitude

$$V_T = |\mathbf{V}| \sin\varphi \tag{10}$$

where V_T is the axial component, $|\mathbf{V}|$ is the velocity magnitude and φ is the helix angle with respect to the horizontal plane, which is given by

$$\varphi = \tan^{-1}\left(\frac{h}{\pi D_H}\right) \tag{11}$$

h corresponds to the vertical bubble displacement that corresponds to a full oscillation cycle and D_H is the helix diameter. In terms of amplitude and frequency of oscillation, one has

$$\varphi = \tan^{-1}\left(\frac{V_T f}{2\pi A}\right) \tag{12}$$

where f is the frequency and A is the amplitude.

Even if there are some investigations that explore analytically the effect of the non-steadiness of the flow in unconfined single rising bubbles, there are still no adequate mathematical descriptions of the bubble oscillatory motion and its perturbation effect on the surrounding liquid. This is because the helix angle determination in Eq. (10) had to be done from empirical data.

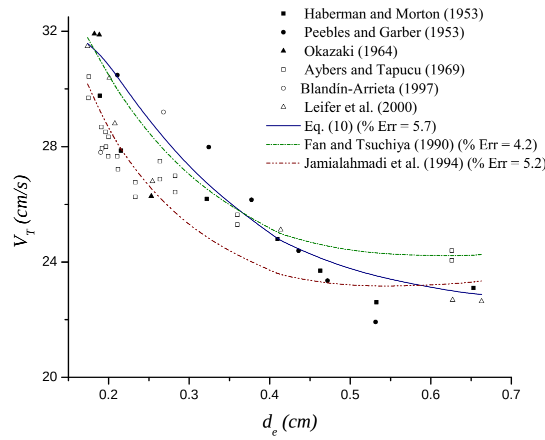

Fig. 4. Comparison between predictions of equations and experimental data for bubble terminal velocity in distilled water ($560 < Re < 1500; 2.3 < We < 5.9; 0.5 < Eo < 5.9$) a 20-28 °C. Markers correspond to experimental data, and lines to predictions from the equations.

There are several investigations that studied the oscillatory motion of single unconfined bubbles (Aybers and Tapucu, 1969; Blandín-Arrieta, 1997; Ford and Loth, 1998; Brucker, 1999; Ellingsen and Risso, 2001; Wu and Gharib, 2002; Shew and Pinton, 2006a,b; Shew *et al.*, 2006; Veldhuis *et al.*, 2008). It has been observed experimentally that the oscillations amplitude $A \sim 0.0035$ m and frequency $f \sim 5$ Hz (Loth, 2010). When these values are substituted into Eq. (12) a correction for the terminal velocity is obtained in Eq. (10). This correction depends on the amplitude, frequency and experimental values for the terminal velocity. The following equation was consequently fitted

$$\varphi = 71.388 - 12.054 d_e \tag{13}$$

Taking into account all the aforementioned facts, Fig. 4 plots the predictions of Eq. (10) considering the helical paths followed by the bubbles. It should be remarked that when $d_e < 0.17$ and $d_e > 0.67$, then $\varphi = 90°$ and the trajectory is rectilinear ($\sin\varphi = 1$). The corrected fit is similar to that of Fan and Tsuchiya (1990) and Jamialahmadi *et al.* (1994) equations, all with a relative average error of about 5 % (absolute).

Experimental data for other liquids than water are scarce. Aqueous solutions and water are most relevant for practical purposes. Even, Eq. (1), along with Eqs. (6) and (8) can also be used to calculate the bubble terminal velocity for pure liquids.

Table 1. Critical equivalent diameter (d_{ec}) for the onset of path instability of rising bubbles in pure liquids at 20-22 °C.

Liquid	d_{ec} (cm) [Eq. (1) with (6) and (8)]	d_{ec} (cm) (Experimental data)	Reference
Pure water	0.17	0.18	Duineveld (1995)
Methanol	0.12	0.13	Hartunian and Sears (1957)
Ethanol	0.16	0.17	Hartunian and Sears (1957)
Benzene	0.12	0.12	Hartunian and Sears (1957)
Silicone oil (DMS-T00)	0.11	0.13	Zenit and Magnaudet (2008)
Silicone oil (DMS-T02)	0.19	0.22	Zenit and Magnaudet (2008)

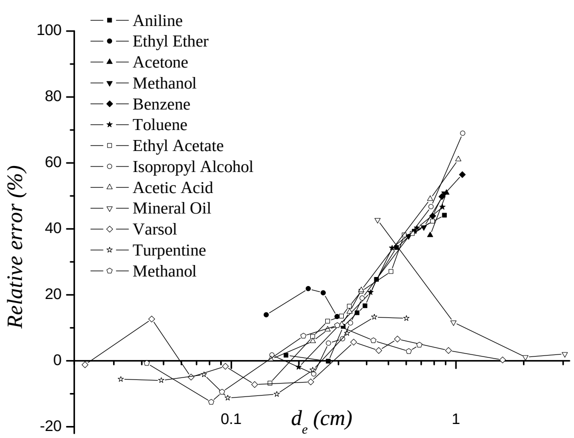

Fig. 5. Relative error (%) for the predictions of the proposed formulation compared to experimental data for various pure liquids.

In Fig. 5 the relative errors for the formulation predictions of this work with respect to experimental data by Haberman and Morton (1953) and Peebles and Garber (1953) are shown. Even if the error values are not as small as for pure water, 72 % of the points have a relative error less than 25 %. These results are similar to those obtained by using Fan and Tsuchiya (1990) and Jamialahmadi *et al.* (1994) equations. With respect to the path instability critical diameter prediction, the formulation of this work is good for bubbles rising in pure liquids of low viscosity, according to the comparison against experimental results, shown in Table 1. For bubbles rising in viscous liquids, it is possible that the oscillatory regime could not be clearly observed, since it can be inhibited by viscous effects. Anyway, more thorough studies for pure liquids, other than water, are still necessary to explain satisfactorily the motion of bubbles and their corresponding regimes.

Conclusions

An equation was obtained for the terminal velocity of bubbles rising in pure liquids, particularly valid for the intermediate and inertial regimes of bubble motion. The formulation uses a weighting of the force balance obtained from boundary layer theory for non-distorted bubbles and an analytical equation coming from a mechanical energy balance. Predictions obtained this way fit better the available experimental data for bubbles rising through pure water (d_e > 0.04 cm), improving the results obtained by other similar formulations that do not require bubble shape geometrical parameters. Oscillatory motion is the only situation that required the introduction of a correction, assuming helical rising paths. In general, the formulation proposed in this work showed to be adequate to predict terminal velocity of rising bubbles in pure liquids. Finally, the local extrema, which bound the onset of the path instability (local maximum) and the point where inertial effects begin to dominate (local minimum), are well predicted by the proposed formulation.

Acknowledgments

The authors respectfully acknowledge the Consejo Nacional de Ciencia y Tecnología (CONACyT - México) for their support through a Graduate scholarship (SBR), as well as through Grant CB-2005-C01-50379-Y.

Nomenclature

A	amplitude of mean oscillation cycle for helical bubble motion (m).
d_e	diameter of a spherical bubble of equivalent volume (m).
d_{ec}	critical equivalent diameter for onset of path instability in bubble motion (m).
D_H	helix mean diameter for spiraling path in bubble motion (m).
E	aspect ratio between minor and mayor axes of oblate ellipsoidal bubbles.
f	frequency of mean oscillation cycle for helical bubble motion (Hz).
g	acceleration of gravity (m/s^2).
h	vertical bubble displacement that corresponds to a mean oscillation cycle for helical bubble motion (m).
$\|\mathbf{V}\|$	magnitude of the bubble motion velocity vector (m/s).
V_T	terminal velocity of single bubbles (m/s).
V_{T1}	bubble rise velocity when viscous effects are important (m/s).
V_{T2}	bubble rise velocity when surface tension effects are important (m/s).

Greek letters

$\Delta\rho$	density difference between liquid and gaseous phases (kg/m^3)
μ_L	dynamic viscosity of liquid phase (Pa s)
ρ_L	density of liquid phase (kg/m^3).
σ	surface tension (N/m)
v	liquid phase velocity due to bubble displacement (m/s).
φ	helix mean angle for spiraling path in bubble motion (°).

Dimensionless numbers

Eo	Eötvos number ($\Delta\rho d_b^2 g/\sigma$).
Re	particle Reynolds number ($\rho_L d_b V_T/\mu_L$).
We	Weber number ($\rho_L d_b V_{T2}/\sigma$).

References

Aybers, N. M. and Tapucu, A. (1969). The Motion of gas bubbles rising through stagnant liquid. *Heat and Mass Transfer 2*, 118-128.

Blandín-Arrieta, J. (1997). *Visualization and Description of Bubble Dynamics* (in Spanish). Chemical Engineering MSc. Thesis, Universidad Autónoma Metropolitana - Iztapalapa, México.

Bozzano, G. and Dente, M. (2001). Shape and terminal velocity of single bubble motion: a novel approach. *Computers and Chemical Engineering 25*, 571-576.

Brucker, C. (1999). Structure and dynamics of the wake of bubbles and its relevance for bubble interaction. *Physics of Fluids 11*, 1781-1796.

Cai, Z., Bao, Y. and Gao, Z. (2010). Hydrodynamic behavior of a single bubble rising in viscous liquids. *Chinese Journal of Chemical Engineering 18*, 923-930.

Celata, G. P., D'Annibale, F., Di Marco, P., Memoli, G. and Tomiyama, A. (2007). Measurements of rising velocity of a small bubble in a stagnant fluid in one and two component systems. *Experimental Thermal and Fluid Science 31*, 609-623.

de Vries, A. W. G., Biesheuvel, A. and van Wijngaarden, L. (2002). Notes on the path and wake of a gas bubble rising in pure water. *International Journal of Multiphase Flow 28*, 1823-1835.

Duangsuwan, W., Tuzun, U. and Sermon, P. A. (2011). The dynamics of single air bubbles and alcohol drops in sunflower oil at various temperatures. *AIChE Journal 57*, 897-910.

Duineveld, P. C. (1995). The rise velocity and shape of bubbles in pure water at high Reynolds number. *Journal of Fluid Mechanics 292*, 325-332.

Ellingsen, K. and Risso, F. (2001). On the rise of an ellipsoidal bubble in water: Oscillatory paths and liquid induced velocity. *Journal of Fluid Mechanics 440*, 235-268.

Fan, L.-S. and Tsuchiya, K. (1990). *Bubble Wake Dynamics in Liquids and Liquid Solid Suspensions*. Butterworth-Heinemann Series in Chemical Engineering.

Ford, B. and Loth, E. (1998). Forces on ellipsoidal bubbles in a turbulent shear layer. *Physics of Fluids 10*, 178-188.

Haberman, W. L. and Morton, R. K. (1953). An experimental investigation of the drag and shape of air bubbles rising in various liquids. *David Taylor Model Basin* Report 802, US Department of the Navy, Washington, DC.

Hadamard, J. S. (1911). Movement permanent lent d'une sphère liquide et visqueuse dans un liquide visqueux. *Comptes Rendus Hebdomadaires des Seances de V Academie des Sciences (Paris) 152*, 1735-1738.

Hartunian, R. A. and Sears, W. R. (1957). On instability of small gas bubbles moving uniformly in various liquids. *Journal of Fluid Mechanics 3*, 27-47.

Jamialahmadi, M., Branch, C. and Muller-Steinhagen, H. (1994). Terminal bubble rise velocity in liquids. *Chemical Engineering Research and Design 72*, 119-122.

Kulkarni, A. A. and Joshi, J. B. (2005). Bubble formation and bubble rise velocity in gas-liquid systems: A review. *Industrial and Engineering Chemistry Research 44*, 5873-5931.

Lehrer, I. H. (1976). A rational terminal velocity equation for bubbles and drops at intermediate and high Reynolds numbers. *Journal of Chemical Engineering of Japan 9*, 237-240.

Leifer, I., Patro, R. K. and Bowyer, P. (2000). A study on the temperature variation of rise velocity for large clean bubbles. *Journal of Atmospheric and Oceanic Technology 17*, 1392-1402.

Levich, V. G. (1962). *Physicochemical Hydrodynamics*. Prentice Hall Englewood Clift.

Loth, E. (2010). *Particles, Drops and Bubbles: Fluid Dynamics and Numerical Methods*. University of Illinois at Urbana-Champaign & University of Virginia. Book Draft for Cambridge University Press. Available on (May 7, 2010): http://www.ae.illinois.edu/l̃oth/CUP/Loth.htm.

Marks, C. H. (1973). Measurements of the terminal velocity of bubbles rising in a chain. *Journal of Fluids Engineering 9*, 17-22.

Mendelson, H. D. (1967). The Prediction of Bubble Terminal Velocities from Wave Theory. *AIChE Journal 13*, 250-253.

Moore, D. W. (1963). The boundary layer on a spherical gas bubble. *Journal of Fluid Mechanics 16*, 161-176.

Moore, D. W. (1965). The velocity of distorted gas bubbles in a liquid of small viscosity. *Journal of Fluid Mechanics 23*, 749-766.

Mougin, G. and Magnaudet, J. (2006). Wake-induced forces and torques on a zigzagging/spiralling bubble. *Journal of Fluid Mechanics 567*, 185-194.

Okawa, T., Tanaka, T., Kataoka, I. and Mori, M. (2003). Temperature effect on single bubble rise characteristics in stagnant distilled water. *International Journal of Heat and Mass Transfer 46*, 903-913.

Okazaki, S. (1964). The Velocity of Ascending air bubbles in aqueous solutions of a surface active substance and the life of the bubble on the same solution. *Bulletin of the Chemical Society of Japan 37*, 144-150.

Peebles, F. N. and Garber, H. J. (1953). Studies on the motion of gas bubbles in liquids. *Chemical Engineering Progress 49*, 88-97.

Rodrigue, D. (2001). Generalized correlation for bubble motion. *AIChE Journal 47*, 39-44.

Ruzicka, M. C., Zahradnik, J., Drahos, J. and Thomas, N. H. (2001). Homogeneus-heterogeneous regime transition in bubble columns. *Chemical Engineering Science 56*, 4609-4626.

Shah, AND. T., Kelkar, B. G., Godbole, S. P., Deckwer, W-D., 1982. Design parameters estimation for bubble column reactors. *AIChE Journal 28*, 353-379.

Shew, W. L. and Pinton, J.-F. (2006a). Dynamical model of bubble path instability. *Physical Review Letters 97*(144508), 1-4.

Shew, W. L. and Pinton, J.-F. (2006b). Viscoelastic effects on the dynamics of a rising bubble. *Journal of Statistical Mechanics: Theory and Experiment* P01009.

Shew, W. L., Poncet, S. and Pinton, J.-F. (2006). Force measurements on rising bubbles. *Journal of Fluid Mechanics 569*, 51-60.

Sanada, T., Sugihara, K., Shirota, M. and Watanabe, M. (2008). Motion and drag of a single bubble in super-purified water. *Fluid Dynamics Research 40*, 534-545.

Talaia, M. A. R. (2007). Terminal velocity of a bubble rise in a liquid column. *Proceedings of the World Academy of Science, Engineering and Technology 22*, 264-268.

Tomiyama, A., Celata, G. P., Hosokawa, S. and Yoshida, S. (2002). Terminal velocity of single bubbles in surface tension force dominant regime. *International Journal of Multiphase Flow 28*, 1497-1519.

Veldhuis, C., Biesheuvel, A. and van Wijngaarden, L. (2008). Shape oscillations on bubbles rising in clean and tap water. *Physics of Fluids 20*(040705), 1-12.

Wellek, R. M., Agrawal, A. K. and Skelland, A. H. P. (1966). Shape of liquid drops moving in liquid media. *AIChE Journal 12*, 854-862.

Wu, M. and Gharib, W. (2002). Experimental studies on the shape and path of small air bubbles rising in clean water. *Physics of Fluids 14*, L49-L52.

Zenit, R. and Magnaudet, J. (2008). Path instability of rising spheroidal air bubbles: A shape controlled process. *Physics of Fluids 20*(061702), 1-4.

Zudin, AND. B. (1995). Calculation of the rise velocity of large gas bubbles. *Journal of Engineering Physics and Thermophysics 68*, 10-15.