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# STABILIZATION OF THIRD-ORDER SYSTEMS WITH POSSIBLE COMPLEX CONJUGATE POLES AND TIME DELAY

# ESTABILIZACIÓN DE SISTEMAS DE TERCER ORDEN CON POSIBLES POLOS COMPLEJOS CONJUGADOS Y TIEMPO DE RETARDO

M.A. Hernández-Pérez, B. del Muro-Cuéllar[1]*, D. Cortés-Rodríguez and I. Araujo-Vargas

*Escuela Superior de Ingeniería Mecánica y Eléctrica, Unidad Culhuacan, Instituto Politécnico Nacional, México D.F., 04430, México.*

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**Abstract**

The stabilization problem of third-order, time-delay unstable linear systems is analyzed. The systems under consideration have one unstable and two stable poles, which may be complex conjugate. Necessary and sufficient conditions to guarantee the stability of the closed loop system by means of a static output feedback are provided. Using such conditions, a predictor scheme that improves the transient system performance is proposed. To illustrate the application of the proposed strategy, it is applied to an unstable continuously stirred tank reactor model. Simulation results are presented.

*Keywords*: delay, complex conjugate poles, predictor, stabilization.

**Resumen**

Este trabajo considera el problema de la estabilización de sistemas lineales con tiempo de retardo de tercer orden, con un polo inestable y dos estables, los cuales pueden ser complejos conjugados. Se presentan las condiciones necesarias y suficientes para asegurar la estabilidad del sistema en lazo cerrado por medio de una retroalimentación estática de la salida. Asimismo, usando el resultado anterior se propone un esquema predictor el cual mejora el desempeño transitorio del sistema. Finalmente, el desempeño de la estrategia de control propuesta es evaluado mediante su aplicación en simulación numérica a un reactor de tanque continuamente agitado inestable.

*Palabras clave*: retardo, polos complejos conjugados, predictor y estabilización.

## 1 Introduction

Delayed systems, also termed dead-time systems, occur in most of the industrial processes (Pierre, 2003). Time delays are present in systems where some kind of mass or energy is transported, for example in distillation columns, heat exchange processes, etc. Sometimes, delays arise due to the own system design, for example due to computational cost of the control algorithm, remote control and communication networks in distributed systems, etc. However, most of the time, delays are introduced by the sensors and/or actuators devices (Liu *et al*., 2005). Particularly, unstable processes with time delay are common in chemical and industrial processes, like liquid storage tanks (Liu *et al*., 2005), continuously stirred tank reactor (CSTR) (Qing-Chang, 2006), etc.

Delays are perhaps one of the main causes of instability and poor performance, producing in general unwanted behavior in dynamical systems. Therefore, stability analysis and controller design of delayed systems deserve special attention. As a consequence, there is great motivation for studying the effects that delays cause in dynamic systems behavior. From the control point of view, time delays introduce complications that must be overcome by designing control strategies. Such strategies must provide both

*Corresponding author. E-mail: bdelmuro@ipn.mx
Tel. (52)56-562058

acceptable performance and stability of closed-loop system.

The simplest approach is to ignore the delay term, i.e., the compensator is designed for the delay-free processes and applied to the original delayed system. Clearly, this method only works for processes with a sufficiently small delay. Other approach consists on the approximation of the delay operator by means of Taylor or Pade series expansions that leads to non-minimum-phase models with rational transfer function representation (Munz *et al.*, 2009). Another solution to control delayed systems is the use of Proportional-Integral (PI) and Proportional- Integral-Derivative (PID) controllers. An example of this technique is the work presented by Nesimioglu and Soylemez, (2010). In this work a first order unstable process with time delay is analyzed, the stability analysis based on Walton and Marshall (1987) is used to calculate all stabilizing values of proportional controllers for those systems. A complete set of PID-controllers for time-delay systems is analyzed in Silva and Bhattacharyya (2005), where different bounds for the stabilization of first-order, dead-time systems were provided.

The effect of time delay can also be compensated by trying to remove the exponential term from the characteristic equation of the process. This strategy tries to eliminate the time delay effects by means of schemes intended to predict the effects of current inputs in future outputs. This technique was introduced by Smith (1957) and the proposed time delay compensator is known as the Smith Predictor (SP). Such technique does not have a stabilization step, this restricts its application to open-loop stable plants. To deal with this disadvantage, some modifications of the SP original structure have been proposed (see for instance (Seshagiri *et al.*, 2007) and (Kawnish and Shoukat, 2012). In Rao and Chidambaram (2006), it was presented an efficient modification to the Smith predictor to control unstable first order systems with time delay, using the direct synthesis method. In this manner, a lead-lag compensator is designed to control the open-loop, second-order unstable delayed process. With a different perspective, Normey-Rico and Camacho (2008, 2009), propose a modification to the original Smith Structure in order to deal with first-order unstable delayed systems. Other studies have been reported, such as that described in (Xian *et al.*, 2005), which proposes a control scheme with two-degree-of-freedom for enhanced control of unstable delay processes.

Using a different approach, some recent works have been devoted to the analysis of stability and stabilization of systems with delay based on approaches of Lyapunov-Krasovskii and Lyapunov-Razumikhin. These results are expressed in terms of algebraic Riccati equations, (Kolmanovskii and Richard, 1999); Qing-Chang, 2006), etc., or linear matrix inequalities, (Fridman and Shaked, 2002; Lee *et al.*, 2004; Mahmoud and Al-Rayyah, 2009), etc. In Michiels *et al.* (2002), a numerical method is employed to shift the unstable eigenvalues to the left half plane by static state feedback, applying small changes to the feedback gain. The same approach is implemented by considering an observer-based strategy; however, stability conditions with respect to time-delay and time constant of the process are not provided.

In del Muro *et al.* (2009), necessary and sufficient conditions for the stabilization of linear systems by static output feedback with one unstable pole, $n$ real stable poles and time delay $\tau$, are provided. In See *et al.* (2010), the problem is also solved by PI-PID controllers.

This work is aimed to give a further step in this topic. Particularly, the static feedback stabilization problem of delayed system with one unstable and two stable poles, which may be complex conjugate, is addressed. Necessary and sufficient conditions for this problem are derived. Once the stabilization conditions obtained, a predictor-controller scheme to improve its transient response is designed. Transient response performance has a core importance in chemical processes. The proposed scheme is then applied to an unstable Continuously Stirred Tank Reactor to give an example of the control strategy in a practical framework.

The paper is organized as follows; Section 2 introduces the problem statement. In Section 3 the proposed control strategy and the necessary and sufficient conditions for the existence of the stabilizing control structure are derived. In Section 4 the example is presented. Finally, some conclusions are given in the last section.

## 2 Problem statement

Consider the following class of Linear Time Invariant systems (LTI), a Single-Input Single-Output (SISO) with delay in the input-output path,

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)}e^{-\tau s} = G(s)e^{-\tau s}, \qquad (1)$$

where,

- $U(s)$ is the input signal,
- $Y(s)$ is the output signal,
- $\tau \geq 0$ is the time delay,
- $N(s)$ and $D(s)$ are polynomials represented in the complex variable 's',
- $G(s)$ is the delay-free transfer function.

Applying an output feedback control strategy of the form,

$$U(s) = [R(s) - Y(s)]Q(s), \tag{2}$$

to (1) the transfer function $Y(s)/R(s)$ can be obtained yielding

$$\frac{Y(s)}{R(s)} = \frac{Q(s)G(s)e^{-\tau s}}{1 + Q(s)G(s)e^{-\tau s}}, \tag{3}$$

Note that the exponential term $e^{-\tau s}$ located at the denominator of this transfer function, leads to a system with an infinite number of poles. Hence, any closed-loop stability properties must be carefully stated.

Hereafter the third-order delayed system with one unstable and two (possibly complex) stable poles, characterized by

$$\frac{Y(s)}{U(s)} = \frac{\alpha}{(s-a)(s^2 - 2\zeta\ omega_n s + \omega_n^2)}e^{-\tau s}, \tag{4}$$

is considered. The parameters $\zeta$ and $\omega_n$ are the damping relation and the undamped natural frequency respectively. Note that when $0 < \zeta < 1$, a couple of complex conjugate poles is present in the system.

### 2.1 Preliminary results

This Section presents some preliminary results useful to obtain the main result of the paper. Consider the unstable first-order delayed system given by,

$$\frac{Y(s)}{U(s)} = \frac{\alpha}{(s-a)}e^{-\tau s}, \tag{5}$$

with $a > 0$. Consider a proportional output feedback,

$$U(s) = R(s) - kY(s). \tag{6}$$

If the time delay $\tau$ is small compared to the unstable time constant $1/a$, then a gain $k$ exists such that the closed loop system, Eq. (7), is stable.

$$\frac{Y(s)}{R(s)} = \frac{\alpha e^{-\tau s}}{s - a + k\alpha e^{-\tau s}}. \tag{7}$$

This idea may formally be stated as:

**Lemma 1** *Considering the delayed system (5) and the proportional output feedback (6), then, a proportional gain k exists such that the closed loop system (7) is stable if and only if*

$$\tau < \frac{1}{a}$$

The proof of Lemma 1 can be easily obtained by considering different approaches, such as the ones given by (Niculescu, 2001) and (Silva and Bhattacharyya, 2005). A simple proof based on a discrete time approach is shown in (Márquez *et al.*, 2010).

## 3 Proposed control strategy

This section presents the necessary and sufficient conditions to stabilize a third-order delayed system (4) by static output feedback. In order to improve the transient response, an observer scheme is proposed.

### 3.1 Stabilizing conditions

**Lemma 2** *Consider the class of third-order characterized by (4), with the static output feedback*

$$U(s) = R(s) - kY(s), \tag{8}$$

*where $R(s)$ is the new reference input.*

*A gain k exists such that the closed loop system (4) and (8),*

$$\frac{Y(s)}{R(s)} = \frac{\alpha e^{-\tau s}}{(s-a)(s^2 + 2\zeta\omega_n s + \omega_n^2) + k\alpha e^{-\tau s}} \tag{9}$$

*is stable if and only if*

$$\tau < \frac{1}{a} - \frac{2\zeta}{\omega_n}.$$

**Proof** The proof is made by using a frequency domain analysis. From the Nyquist stability criteria, the system is stable if $N + P = 0$, where $P$ is the number of poles in the right half plane 's' and $N$ is the number of clockwise rotations to the (-1,0) point. $N$ is negative if rotations are counterclockwise in the Nyquist diagram.

First, let us to analyze the first order unstable delayed system given by (5). Note that in this case $P = 1$, hence there exists a gain k that stabilizes the system if $N = -1$, i.e., if a there is a counterclockwise rotation to the point (-1,0).

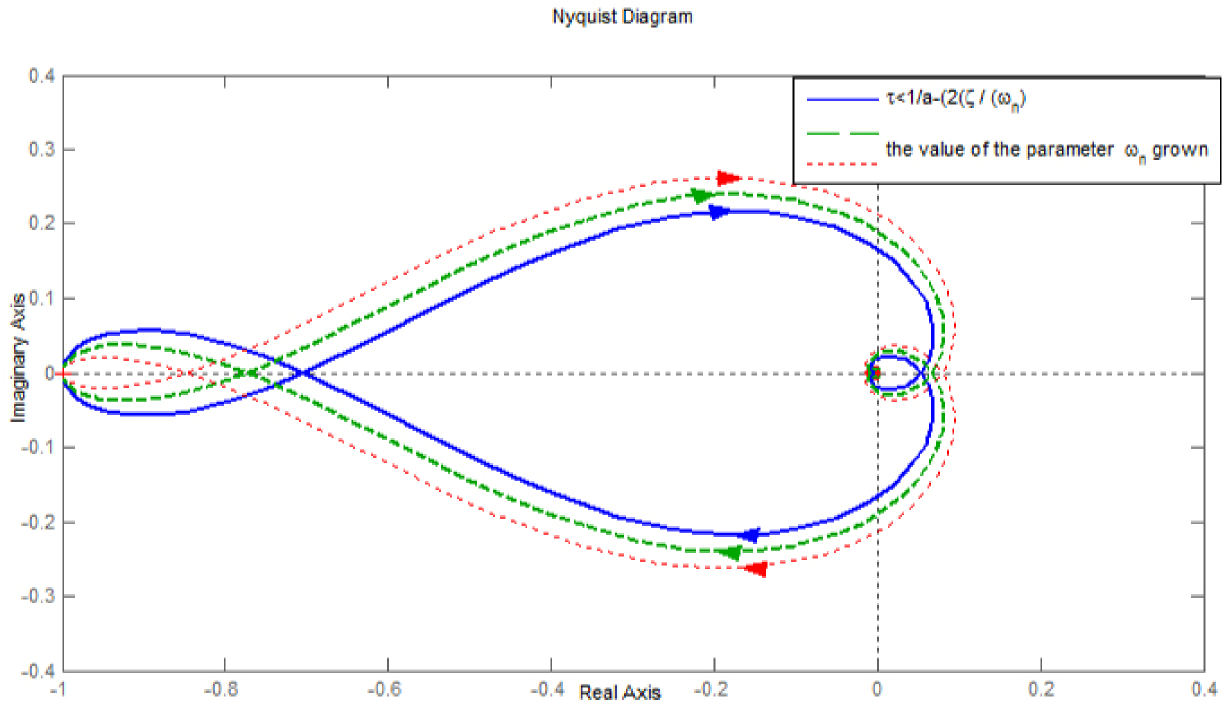


Fig. 1. Nyquist diagram when $\beta$ grows.

Considering a frequency analysis, the phase expression in the frequency domain $\omega$ for (5) is given by,

$$\langle G(j\omega)\rangle = -\left(180^{\circ} - \tan^{-1}\left(\frac{\omega}{a}\right)\right) - \omega\tau.$$

For this reason, condition $\tau < 1/a$ is equivalent to say that $\langle G(j\omega)\rangle = -180^{\circ}$ when $\omega \approx 0$, i.e. the Nyquist diagram starts (for frequencies near to zero) with an angle greater than $-180^{\circ}$. This means that there is a counterclockwise rotation to the point (-1,0). If the point (-1,0) is outside the loop, this can be fixed by adjusting the $k$ value.

Let us now to consider the system (4). The phase expression in the frequency domain $\omega$ is given by

$$\langle G(j\omega)\rangle = -\left(180^{\circ} - \tan^{-1}\left(\frac{\omega}{a}\right)\right) - \omega\tau - \tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right).$$

Define $\beta = \tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$. Since the Nyquist condition is the same (a counterclockwise rotation to the point -1), if the condition $\tau < 1/a$ is satisfied and $\beta$ is small enough, then a gain $k$ exists that stabilizes the system. As the parameter $\beta$ grows, the loop that forms the counterclockwise rotation to the point (-1,0) decreases to extinction (the Nyquist diagram then starts with angle smaller than -180°) as it is illustrated in Fig. 1. Considering that for small frequencies $\tan^{-1}(\omega) \approx \omega$, and taking into account that $\langle G(j\omega)\rangle - 180^{\circ}$, it can be concluded that solving for $\tau$, the stability condition for the system (4) is

$$\tau < \frac{1}{a} - \frac{2\zeta}{\omega_n}.$$

The following corollary provides a useful procedure to compute the parameter k involved in the control scheme.

**Corollary 1** *Consider the system (4). Suppose there exists a gain k such that the corresponding closed loop systems is stable, i.e., the condition $\tau < 1/a - 2\zeta/\omega_n$ holds. Then the family of k stabilizing the closed loop system is given by*

$$k_1(\omega_(c_1)) < k < k_2(\omega_(c_2)),$$

*where*

$$k_{1,2}(\omega_c) = \sqrt{(\omega_c^2 + a^2)(\omega_c^4 + 2\omega_n^2\omega_c^2(2\zeta^2 - 1) + \omega_n^4)};$$

*with $\omega_{c_1} = 0$ and $\omega_{c_2}$ satisfying,*

$$\tan^{-1}\left(\frac{\omega_{c_2}}{a}\right) - \omega_{c_2}\tau - \tan^{-1}\left(\frac{2\zeta\frac{\omega_{c_2}}{\omega_n}}{1-\left(\frac{\omega_{c_2}}{\omega_n}\right)^2}\right) = 0.$$

**Proof** Assuming that $\tau < \frac{1}{a} - \frac{2\zeta}{\omega_n}$, and taking into account that the magnitude expression is,

$$M_G(j\omega) = k\frac{1}{\sqrt{(\omega_c^2 + a^2)(\omega_c^4 + 2\omega_n^2\omega_c^2(2\zeta^2 - 1) + \omega_n^4)}},$$

Note that the magnitude decreases monotonically from $k$ to 0. Hence, the phase will first increase from $-180^o$ and then it will decrease back to $-180^o$. This implies that the phase intersects the negative real axis, i.e. $\langle G(j\omega)\rangle = -180^o$, for some positive frequency. In order to have a counterclockwise rotation of (-1,0), this intersection should lie between -1 and 0, that is, the range of $k$ must be between two values or equivalently $k_1(\omega_{c_1}) < k < k_2(\omega_{c_2})$, with

$$k_1(\omega_{c_1}) = \sqrt{(\omega_{c_1}^2 + a^2)(\omega_{c_1}^4 + 2\omega_n^2\omega_{c_1}^2(2\zeta^2 - 1) + \omega_n^4)},$$

with the crossover frequency $\omega_{c_1} = 0$, and

$$k_2(\omega_{c_2}) = \sqrt{(\omega_{c_2}^2 + a^2)(\omega_{c_2}^4 + 2\omega_n^2\omega_{c_2}^2(2\zeta^2 - 1) + \omega_n^4)},$$

$\omega_{c_2}$ satisfying

$$\tan^{-1}\left(\frac{\omega_{c_2}}{a}\right) - \omega_{c_2}\tau - \tan^{-1}\left(\frac{2\zeta\left(\frac{\omega_{c_2}}{\omega_n}\right)}{1-\left(\frac{\omega_{c_2}}{\omega_n}\right)^2}\right).$$

## *3.2 Conditions for the existence of the predictor*

In many practical applications, is not enough to apply a static output feedback, because the achieved performance is very poor. In particular it may be interesting to obtain a system with a fast settling time with no oscillations even if the open loop system has complex conjugate poles. For that reason the predictor-controller scheme depicted in Fig. 2 is proposed.

Consider the third order delayed system (4). A state space representation, controllable and observable, is given by:

$$\begin{aligned}\dot{x} &= Ax(t) + Bu(t)\\ y(t+\tau) &= Cx(t).\end{aligned} \tag{10}$$

**Lemma 3** *Consider the predictor scheme shown in Fig. 2. There exists a gain k such that* $\lim_{t\to\infty}[\hat{x}(t) - x(t)] = 0$ *if and only if* $\tau < 1/a - (2\zeta)/\omega_n$.

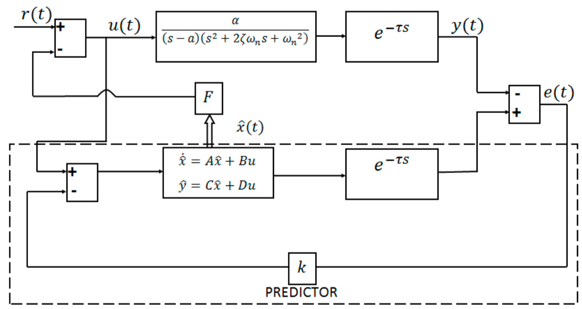


Fig. 2. Predictor-controller scheme.

**Proof** For convenience, this demonstration is performed in state space. Consider the predictor shown in Fig. 2, from (4) a controllable an observable state space representation can be obtained as,

$$\begin{aligned}\dot{x} &= Ax(t) + Bu(t)\\ y(t) &= Cx(t-\tau).\end{aligned} \tag{11}$$

When a static output feedback k is applied the system (11) becomes

$$\dot{x} = Ax(t) + Bu(t) - BkCx(t-\tau). \tag{12}$$

According to Lemma 2, the system can be stabilized if and only if $\tau < 1/a - 2\zeta/\omega_n$. Consider now the predictor shown in Fig. 2, then the system dynamics can be described as follows:

$$\begin{aligned}\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix} &= \begin{bmatrix}A & 0\\ 0 & A\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}\\ &+ \begin{bmatrix}0 & 0\\ Bk & -Bk\end{bmatrix}\begin{bmatrix}x(t-\tau)\\ \hat{x}(t-\tau)\end{bmatrix} + \begin{bmatrix}B\\ B\end{bmatrix}u(t)\\ \begin{bmatrix}y(t)\\ \hat{y}(t)\end{bmatrix} &= \begin{bmatrix}C & 0\\ 0 & C\end{bmatrix}\begin{bmatrix}x(t-\tau)\\ \hat{x}(t-\tau)\end{bmatrix},\end{aligned} \tag{13}$$

where $\hat{x}(t)$ is the estimate of $x(t)$. Define the prediction error as $e_x(t) = \hat{x}(t) - x(t)$. Therefore, it is possible to describe the behavior of the error signal as,

$$\dot{e}_x(t) = \dot{\hat{x}}(t) - \dot{x}(t) = A(\hat{x}(t) - x(t)) - BkC(\hat{x}(t-\tau) - x(t-\tau)). \tag{14}$$

The error dynamics results:

$$\dot{e}_x(t) = Ae_x(t) - BkCe_x(t-\tau). \tag{15}$$

Note that error dynamics given by (15) is the same dynamics that occurs in the system predictor given by (12). Hence there exist a gain $k$ such that $\lim_{t\to\infty}[\hat{x}(t) - x(t)] = 0$ if and only if $\tau < 1/a - 2\zeta/\omega_n$.

### 3.3 Predictor-controller scheme

Considering the control scheme shown in Fig. 2, and the control action given by

$$u(t) = r(t) - F\hat{x}(t).$$

Based on the previous Lemmas, the following Theorem can be stated without needing further proof.

**Theorem 1** *There exist gains k and F such that the Predictor-Controller scheme shown in Fig. 2 is stable if and only if* $\tau < 1/a - 2\zeta/\omega_n$.

**Note 1** *Since the system is controllable, F can be selected such that relocates the poles of the original system in positions where the stability is preserved, ensuring adequate performance.*

**Note 2** *Another way of presenting the main result of this work, is to see the second-order subsystem as a complex conjugate form, that is,*

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{(s + b + jb_i)(s + b - jb_i)},$$

where $b$ is the real part and $b_i$ is the imaginary part of complex conjugate poles. Then, the system (4) may be rewritten as,

$$\frac{Y(s)}{U(s)} = \frac{\alpha}{(s - a)(s + b + jb_i)(s + b - jb_i)} e^{-\tau s}.$$

Applying a static output feedback, the closed-loop system becomes,

$$\frac{Y(s)}{R(s)} = \frac{\alpha e^{-\tau s}}{(s - a)(s + b + jb_i)(s + b - jb_i) + k\alpha e^{-\tau s}}.$$

Then, there exists $k$ such that the closed loop system is stable if and only if

$$\tau < \frac{1}{a} - \frac{2b}{b^2 - b_i^2},$$

where $\zeta\omega_n = b$ and $\omega_n^2 = b^2 + b_i^2$.

Finally, the range of gains for $k_1(\omega_{c_1}) < k < k_2(\omega_{c_2})$ is given by,

$$k_1(\omega_{c_1}) = \sqrt{(\omega_{c_1}^2 + a^2)(\omega_{c_1}^4 + 2\omega_{c_1}^2(2b^2 - b_i^2) + (b^2 + b_i^2)^2)},$$

and

$$k_2(\omega_{c_2}) = \sqrt{(\omega_{c_2}^2 + a^2)(\omega_{c_2}^4 + 2\omega_{c_2}^2(2b^2 - b_i^2) + (b^2 + b_i^2)^2)},$$

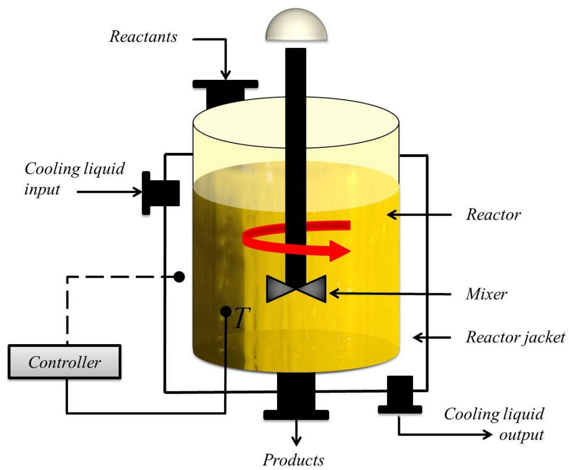


Fig. 3. Unstable Continuously Stirred Tank Reactor (CSTR).

## 4 Example: Application to an unstable continuously stirred tank reactor (CSTR) by numerical simulation

In this section the proposed control strategy is applied to stabilize a chemical reactor system. Specifically, a Continuously Stirred Tank Reactor (CSTR) is considered. The CSTRs are widely used in the chemical process industry. Common examples where this type of reactor is used are the polymerization for the production of plastics and paints, the production of sodium acetate for the formation of soaps, among others.

In most industrial processes, there exist factors that might affect the system performance. For instance, the time delay to take a measurement of temperature or concentration of the reactant in the CSTR. In particular, in this example the time delay is presented in measuring the concentration of reactant $A$.

The purpose of measuring the concentration in the reactant $A$ is to manipulate the speed of the chemical reaction and, thereby, improve the selectivity of the reaction system.

The following example has been taken from Bequette (2003). This model is particularly applied to the chemical process of a Propylene Glycol reactor system, and a general description of the mathematical model is presented below.

In the CSTR, an exothermic reaction occurs, $A \rightarrow B$. To remove the reaction heat, the reactor is surrounded by a jacket (coolant jacket) through which flows a coolant. Often, the heat transfer fluid is pumped through agitation nozzles that circulate the fluid through the jacket at high velocity, as it is shown in Fig. 3.

Preliminary considerations for modeling:

- Heat losses are negligible.
- The thermodynamic properties, densities and heat capacities of the reactants and the products are constants.
- Perfect blend in Reactor.
- Uniform temperatures in both chambers.

Given all these considerations, the variables and parameters are now defined for the plant or process under study.

Independent variables (input variables):

- Product flow $A$: $F$.
- Flow cooling liquid (reactor jacket): $F_{jf}$.

Dependent variables (output variables):

- Concentration of the product $A$: $c_A$.
- Temperature in the Reactor: $T$.
- Temperature in the cooling liquid (reactor jacket): $T_j$.

Measurable disturbances:

- A product concentration $A$ in the reactor input: $C_{Af}$.
- Input temperature product $A$: $T_f$

System parameters are listed in Table 1.

Values at the operating point are shown in Table 2.

Differential equations of the process shown in Fig. 3 are characterized as follows:

Table 1: System parameters of CSTR

| Description | Variables | Values | Units |
|---|---|---|---|
| Volume of the reactor | $V$ | 85 | ft$^3$ |
| Activation energy | $-\Delta E$ | 32,400 | Btu/lbmol |
| Heat transfer coefficient | U | 75 | Btu/hr$^o$F |
| Heat of Reaction | $-\Delta H$ | 39,000 | Btu/lbmol PO |
| Heat transfer area | $A$ | 88 | ft$^2$ |
| Frequency factor | $k_0$ | 16.96 $\times 10^{12}$ | hr$^{-1}$ |
| Ideal gas constant | $R$ | 1.987 | Btu/lbmol$^o$F |
| Volume cooling liquid in the chamber | $V_j$ | 21.25 | ft$^3$ |
| Heat capacity on reactor with density of the reagent | $pc_p$ | 53.25 | Btu/ft$^{3o}$F |
| Density of the cooling liquid with heat capacity of the cooling liquid | $p_{-j}c_{-pj}$ | 55.6 | Btu/ft$^{3o}$F |

Table 2: Values at the operating point of CSTR

| Description | Variables | Values | Units |
|---|---|---|---|
| A product concentration A in the reactor input | $C_{Af}$ | 0.132 | lbmol/ft$^3$ |
| Concentration of the product A | $C_A$ | 0.066 | lbmol/ft$^3$ |
| Product Flow A | $F$ | 340 | ft$^3$/hr |
| Flow cooling liquid (reactor jacket) | $F_{jf}$ | 28.75 | ft$^3$/hr |
| Temperature in the reactor | $T$ | 101.1 | $^o$F |
| Temperature in the cooling liquid (reactor jacket) | $T_j$ | 55 | $^o$F |
| Input temperature product A | $T_j$ | 60 | $^o$F |
| Cooling liquid temperature at the input | $T_{jf}$ | 50 | $^o$F |

1. Balance of the mass on component $A$ is

$$V\frac{dC_A}{dt} = FC_{Af} - FC_A - Vr_A,$$

with

$$r_A = k_0 e^{-\frac{\Delta E}{RT}} C_A,$$

where $r_A$ is the rate of reaction per unit volume, $k_0$ is the frequency factor, $\Delta E$ is the activation energy and $R$ is the ideal gas constant, such that,

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{Af} - C_A) - k_0 e^{-\frac{\Delta E}{RT}} C_A.$$

2. Energy balance in the reactor,

$$\frac{dT}{dt} = \frac{F}{V}(T_f - T) + \frac{-\Delta H}{RT} k_0 e^{-\frac{\Delta E}{RT}} C_A - \frac{UA}{Vpc_p}(T - T_j).$$

3. Energy balance in the cooling liquid,

$$\frac{dT_j}{dt} = \frac{F_{jf}}{V_j}(T_{jf} - T_j) + \frac{UA}{V_j p_j c_p j}(T - T_j).$$

To linearize the differential equations, a Taylor series approximation, or a Jacobian matrix linear transformation around the operating point of Table 2 may be used (Bequette, 2003).

Let consider the jacket temperature as the manipulated variable and the concentration of the CSTR as the controlled variable. Linearization around this steady-state operating point yields the following transfer function model, assuming a measurement time delay of 0.25 hr.

$$\frac{C_A(s)}{T_{jf}(s)} = \frac{0.0646}{(s - 1.1772)(s^2 + 10.51s + 29.26)} e^{-.25s}.$$

For the current example, the parameters of the system are $a = 1.1772$, $\zeta = 0.97$, $\omega_n = 5.4$ and $\tau = 0.25$.

Since $0.25 < 1/1.1772 - (2 * 0.97)/5.4 = 0.49$, the stability condition given in Theorem 1 is satisfied, such that this system may be stabilized by the predictor-controller scheme shown in Fig 2.

According to Corollary 1 the range of values for the stabilizing parameter $k$ is,

$$531.915 < k < 925.926.$$

Selecting $k = 600$, the corresponding Nyquist diagram is shown in Fig. 4.

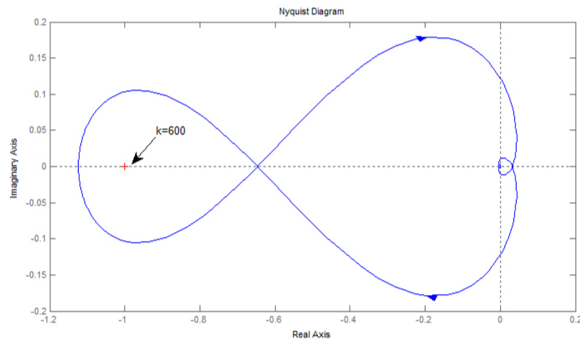


Fig. 4. Nyquist diagram

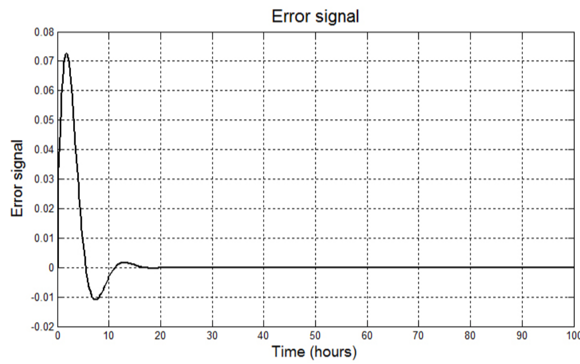


Fig. 5. Error signal $\hat{y}(t) - y(t)$.

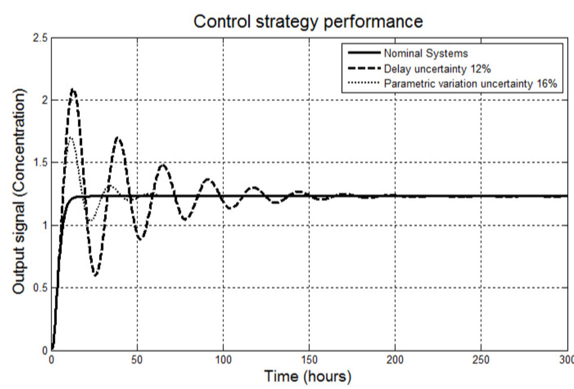


Fig. 6. Control performance of CSTR.

Using the predictor-controller scheme proposed in this work, the output error signal $\lim_{t\to\infty}[\hat{y}(t) - y(t)] = 0$ as shown in Fig. 5.

Hence this system can be stabilized and the poles of the system can be relocated. The gain vector F that relocates the poles at {-1,-3,-5} is F=[-3.33 -5.88 40.43].

Fig. 6 illustrates the simulated performance of the observer-based controller for a unit step reference. The continuous line denotes the performance of the nominal system.

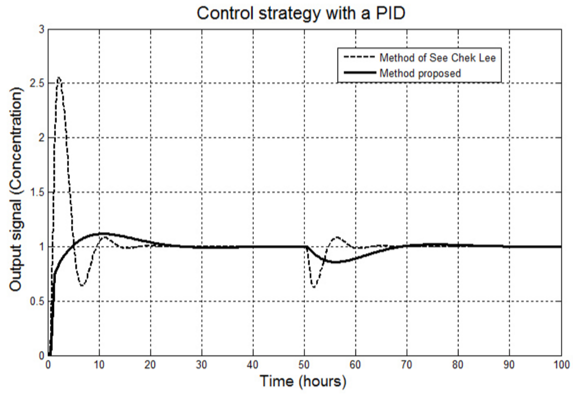


Fig. 7. Control performance of CSTR.

The dashed line shows the performance for the system with an uncertainty of 12% in the delay operator.

Note that the system preserves the stability despite having some fluctuations in the response, caused by variations in parameters up to 16% of its nominal value (see Fig. 6).

The control structure can be complemented with a more sophisticated strategy, like a Proportional-Integral-Derivative (PID) action, in order to obtain step disturbance rejection and step reference tracking. Following this approach, the tuning of the PID controller of the reactor was based in the Ziegler and Nichols tuning rules, ignoring the delay term. The parameters are set to $k_p = 108$, $k_i = 0.3$ and $k_d = 0.8$.

In order to evaluate the performance of the proposed methodology, the results were compared with a PID controller designed with the methodology proposed by Lee Chek See (See *et al*., 2010). According to this methodology, the parameters are: $k_p = 69$, $k_i = 0.3973$ and $k_d = 0.4$.

In Fig. 7, the response of the closed loop system for the two control strategies are shown. A unit step and a step disturbance of magnitude -2 units, acting at the instant of 50 hours are the system inputs.

It can be observed from Fig. 7 that a better performance of the controller proposed in this paper is observed when compared with (See *et al*., 2010).

# Conclusions

Unstable systems with time delay are commonly found in industrial processes, where time delay usually adds complications for its study, and makes the system stabilization a challenge. In this paper are presented the necessary and sufficient conditions for the stabilization, by static output feedback, of a class of unstable third-order systems with possible complex conjugate poles and time delay. A predictor-controller scheme to stabilize the system was proposed and a better performance was achieved. Explicit necessary and sufficient conditions were stated for the existence of the proposed scheme. The control strategy presented in this work was shown to be simple and easy to implement, providing adequate performance despite parameter variations and nonzero initial conditions. An Unstable Continuously Stirred Tank Reactor was used to verify the performance of the proposed strategy using numerical simulation. In this example, a PID controller was added to illustrate that the proposed control technique allows step reference tracking and rejection of step disturbances.

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