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ROBUST DISCRETE OUTPUT REGULATION OF NONLINEAR PROCESSES WITH TIME DELAYED MEASUREMENTS

REGULACIÓN ROBUSTA DISCRETA DE LAS SALIDAS DE PROCESOS NO LINEALES CON MEDICIONES CON TIEMPO DE RETARDO

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Abstract
This paper addresses the robust discrete output regulation problem of nonlinear processes with sampled delayed measurements by extending the robust regulation theory based on system immersion and manifold invariance. The proposed robust regulator consists of a robust model based nonlinear controller and an updating procedure that enables the calculation of the control effort from the delayed measurements even at the intersampling periods. The performance of the proposed discrete control scheme is evaluated under a highly uncertain scenarios by means of a study case that aims at the regulation of the pollutant agents concentration in a biological wastewater treatment process in the face of load changes and time varying set-point values. Numerical and experimental results demonstrate that the proposed control approach is able to track desired set-point trajectories under both the influence of modeling and parametric uncertainties and uncertain load disturbances.
Keywords: robust discrete control, nonlinear systems, time delay compensation, reactor control, immersion and manifold invariance, anaerobic digestion.

Resumen
El presente trabajo trata sobre el problema de la regulación robusta discreta de procesos no lineales utilizando muestreo con retardo mediante la extensión de la teoría de la regulación robusta basada en la inmersión del sistema y las variedades invariantes. El regulador robusto propuesto en este trabajo consiste de un controlador no lineal basado en el modelo y un procedimiento de actualización que permite calcular la señal de control a partir de mediciones con retardo que pueden ser incluso fracciones del tiempo de muestreo. El desempeño del esquema de control discreto propuesto se evalúa en los escenario mas inciertos a través de un caso de estudio que trata de la regulación de agentes contaminantes en un proceso de tratamiento biológico de aguas residuales en presencia de cambios de carga y referencias variables. Los resultados numéricos y experimentales demuestran que esquema de control propuesto es capaz de seguir las trayectorias de referencia deseadas bajo la influencia de incertidumbres paramétricas y perturbaciones en el flujo de alimentación.
Palabras clave: control robusto discreto, sistema no lineal, retardo, control de reactores, inmersiones y variedades invariantes, digestión anaerobia.

1 Introduction

The conventional way to cope the control of nonlinear dynamical time delayed systems is to neglect the presence of dead times and design the controller on the basis of the resulting linear/nonlinear ordinary differential equations by employing standard control methods. However, it is well-known that such an approach may pose unacceptable limitations on the achievable control quality and may cause serious problems in the behavior of the closed-loop system including poor performance (e.g., sluggish response and oscillations) and instability (Frankl'm et al., 1994; Hernández-Pérez et al., 2013). As a consequence, advances in the analysis and control of linear systems with time delays, has been extended to the nonlinear case to deal with stabilization problems, tracking of reference signals and disturbance decoupling (Lee and

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Dianat, 1981; Kravaris and Wright, 1989; Isidori and Byrnes, 1990a; Velasco et al., 1993; Shyu and Yan, 1994). Other approaches include the extension of the basic input-output linearizing control techniques combined with output predictors to handle the effect of time delays (Henson and Seborg, 1994). In the case of robust stabilization, Lee et al. (1994b,a) have applied H_∞ techniques, while Shyu and Yan (1994) have used a variable structure controller to guarantee the system stability but showed some limitations when dealing with the tracking problem because the input required a non-constant zero error submanifold. Moreover, Wu and Chou (1996) designed a robust controller for input delayed systems using input-output linearization techniques which only rendered an attenuated zero error.

In the las two decades, an innovative method has gained special attention to design asymptotic stabilizing and robust control schemes for nonlinear systems. It relies upon the notions of system immersion and manifold invariance where an observable dynamic system able to reproduce the dynamic behavior of the exosystem, allows generating all possible steady state inputs for all admissible values of the system parameters. Conditions to solve the robust regulation problem in terms of immersions have been stated in the research reports of Huang and Isidori (Huang and Lin, 1994; Huang, 1995; Huang and Chen, 2004; Isidori, 1995; Serrani et al., 2001). For instance, Isidori (1995) has shown that when a linear immersion is found, then the problem can be solved by using a linear controller, and Huang (2001) found that a linear immersion exists only when the steady state input is given by a polynomial of the exosystem states.

In this contribution, an error feed-back discrete controller based on system immersion and manifold invariance tools is proposed to solve the robust discrete regulation problem for a class of nonlinear processes with delayed measurements in the face of both modeling and parametric uncertainties and uncertain load disturbances. For this purpose, the robust model-based nonlinear control theory is extended to allow time delay compensation (García-Sandoval et al., 2007), which consists of an updating procedure that takes advantage of the measurements as they become readily available. The performance of the proposed scheme is tested in a study case under a highly uncertain environment by using sampled delayed measurements that focuses on the rejection of uncertain load disturbances during the regulation of the organic pollution level in an actual wastewater treatment experimental bioprocess. The paper is outlined as follows. First, the robust output discrete regulation problem is stated and some basic results of the robust regulation theory are reviewed. Then, the robust discrete regulator is proposed. Finally, the controller performance and robustness are evaluated for the proposed case studies under different operating conditions and uncertain scenarios.

2 On the discrete regulation problem for a class of nonlinear systems with delayed measurements

2.1 Control problem statement

Let us consider the following nonlinear system

$$\dot{x}(t) = f(x(t), u(t), w(t), \lambda), \quad (1)$$
$$\dot{w}(t) = s(w(t)) \quad (2)$$
$$e(t) = h(x(t-\tau), w(t-\tau), \lambda), \quad (3)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are respectively, the state and input system variables, subject to disturbances and/or references signals $w(t)$, defined in a neighborhood of the origin of $\mathbb{R}^s$, whose dynamics are described by an autonomous exosystem given by eq. (2), $e(t) \in \mathbb{R}$ represents a delayed output tracking error between the system output and the reference signal, and $\tau = \varepsilon\delta$, with $\varepsilon \in (0, 1]$ is a constant delay that may be a fraction or equal to the sampling period, δ. Finally, $\lambda \in \mathbb{R}^q$ is an uncertain parameter vector which may take values in a neighborhood $\wp \subset \mathbb{R}^q$. Here, it is assumed that mappings f, s and h are smooth in their arguments and that $f(0, 0, 0, 0) = 0$, $s(0) = 0$ and $h(0, 0, 0) = 0$.

Now, let us introduce the following assumptions which will be useful in the control problem statement.

Assumption 1. *Output (3) is sampled with a period δ at instants $t = k\delta$, for $k = 0, 1, 2, \ldots$, but its actual measurement, due to preparation, characterization and analysis time period, is available at time $t_k = k\delta + \tau$, until after a time delay τ that may be a fraction of the sampling period, i.e., $\tau = \varepsilon\delta$, where $\varepsilon \in (0, 1]$.*

Assumption 2. *For the tracking problem, it is assumed that there exist inputs at any time such that the equilibrium point, $w = 0$, is stable in the sense of Lyapunov, and the eigenvalues of $S = \left.\frac{\partial s}{\partial w}\right|_{w=0}$ lie on the imaginary axis.*

Then, the *Nonlinear Output Delayed Discrete Robust Regulation Problem* (NODDRRP) consists in finding, if possible, a feedback dynamic discrete controller with sampling period δ, such that, for all admissible parameter values λ, the following conditions are satisfied

Stability: The solution of the closed-loop system, without disturbances (i.e., with $w = 0$) but with parametric variations at the sampling instants goes asymptotically to zero.

Regulation: For each initial condition in a neighborhood of the origin, the solution of the closed-loop system, with disturbances and parametric variations, guarantees that $\lim_{t\to\infty} e(t) = 0$.

2.2 Fundamental results in the nonlinear robust regulation problem

Delli-Priscoli et al. (1997) have proposed a local solution to the Robust Regulation Problem stated in terms of the existence of the nonlinear mappings $x_{ss}(t) = \pi(w(t), \lambda)$ and $u_{ss}(t) = \gamma(w(t), \lambda)$ which solve the equations

$$\frac{\partial \pi(w,\lambda)}{\partial w} s(w) = f(\pi(w,\lambda), \gamma(w,\lambda), w, \lambda) \quad (4)$$

$$0 = h(\pi(w,\lambda), w, \lambda) \quad (5)$$

for all admissible values of $\mu \subset \wp$, with $\pi(0,\lambda) = 0$ and $\gamma(0,\lambda) = 0$, both defined in a neighborhood of the origin of $(w,\lambda) = (0,0)$ and, in particular for linear immersions, such that, there exists a set of real numbers $a_0, a_1, \ldots, a_{r-1}$ satisfying

$$L_s^r \gamma(w,\lambda) = a_0 \gamma(w,\lambda) + a_1 L_s \gamma(w,\lambda) + \cdots + a_{r-1} L_s^{r-1} \gamma(w,\lambda) \quad (6)$$

where $L_s^k \gamma(w,\lambda)$ stands for the Lie derivative defined as $L_s^k \gamma(w,\lambda) = \left[\frac{\partial L_s^{k-1}\gamma(w,\lambda)}{\partial w}\right] s(w)$; $k \geq 1$ with $L_s^0 \gamma(w,\lambda) = \gamma(w,\lambda)$.

Equation (6) can be recast into the following linear dynamic system (Isidori, 1995)

$$\dot{z} = \Phi z, \quad (7)$$

$$\gamma(w,\lambda) = Hz. \quad (8)$$

which may be used to generate the steady state input, regardless of the values of λ. Here

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{r-1} \end{pmatrix} \quad (9)$$

$$H = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}_{1\times r}$$

Thus, this system may be viewed as an immersion of the exosystem (2) into a linear observable system.

Remark 3. Eqs. (4) and (5) are known as the Francis-Isidori-Byrnes equations (FIB) (Isidori and Byrnes, 1990b) and $x_{ss}(t) = \pi(w(t), \lambda)$ represents the steady state zero output submanifold whereas $u_{ss} = \gamma(w(t), \lambda)$ is the steady state input which makes invariant such submanifold. Moreover, $u(t) = u_e(t) + u_{ss}(t)$ (here u_e allows to achieve the zero output submanifold usually by means of an error or state feedback, while u_{ss} is generated by using immersion (7)-(8)).

In the particular case of dynamical systems with no time delay ($\tau = 0$) where the output error is measured at each sampling time δ, Castillo-Toledo and Di'Gennaro (2002) found that in order to eliminate the intersample rippling in the output tracking error, it is necessary to reconstruct the continuous internal model (7)-(8) from its discrete time realization

$$z_d(k+1) = e^{\Phi\delta} z_d(k)$$

where subindex d represents the discrete time realization of z, i.e., $z(k\delta) = z_d(k)$.

One way to reconstruct such a model is through an *exponential holder* which is an analog device that produces a continuous signal from a discrete one and may be obtained as follows.

Let us consider the solution of eqs. (7)-(8) given by

$$z(t) = e^{\Phi(t-t_0)} z(t_0)$$

$$u_{ss}(t) = Hz(t).$$

Setting $t = k\delta + \theta$, $\theta \in [0,\delta)$, and $t_0 = k\delta$, one gets

$$z(k\delta + \theta) = e^{\Phi\theta} z(k\delta)$$

$$u_{ss}(k\delta + \theta) = Hz(k\delta + \theta),$$

and as a consequence

$$u_{ss}(t) = He^{\Phi\theta} z_d(k), \quad t \in [k\delta, (k+1)\delta), \quad \theta \in [0,\delta), \quad k = 1,2,3,\ldots \quad (10)$$

which describes *exactly* the steady-state input not only at the sampling time instants but also at the intersampling period. It is worth mentioning that signal θ in the exponential holder is a periodic sawtooth signal(Castillo-Toledo and Obregon-Pulido, 2003).

For the particular case of $\tau = 0$, a robust regulator that solves the Robust Regulation Problem by using the exponential holder (10) was described by Castillo-Toledo and Di'Gennaro (2002) and it is given by

$$\xi_d(k+1) = (A_{d0} + B_{d0}K_d - G_{d1}C_{d0})\,\xi_d(k) + G_{d1}e_d(k) \quad (11)$$

$$\zeta_d(k+1) = -G_{d2}C_{d0}\xi(k) + \Phi_d\zeta_d(k) + G_{d2}e_d(k) \quad (12)$$

$$u(t) = K_d\xi_d(k) + He^{\Phi\theta}\zeta_d(k) \qquad \theta \in [0,\delta), \quad t = k\delta + \theta,\ k = 0,1,2,\ldots \quad (13)$$

where $A_{d0} = e^{A_0\delta}$, $B_{d0} = \int_0^\delta e^{A_0\varsigma}d\varsigma B_0$ and $\Phi_d = e^{\Phi\delta}$, while matrices K_d, G_{d1} and G_{d2} are feedback gains computed as described by Castillo-Toledo and Di'Gennaro (2002). Here, eq. (11) represents an observer of the error at sampling time $t = k\delta$ given by $\xi_d(k) = x(k\delta) - \pi(w(k\delta),\lambda)$, while eq. (12) is the input steady state observer which uses the discrete version of immersion (7). Finally, the input (13) is composed of a discrete error feedback and a continuous estimated input steady state which uses the exponential holder. The robustness of controller (11)-(13) relies on the immersion (7)-(8) which is able to generate any input $\gamma(w,\lambda)$ simply by knowing the structural behavior of the exosystem w without the knowledge of the exact value of the uncertain parameters λ. The resulting robustness property, better known as the structural robustness, guarantees that the conditions of stability and regulation hold in a certain neighborhood while its actual boundaries are unknown.

2.3 *Robust discrete regulator*

The robustness of controller (11)-(13) is only guaranteed when no input delay is present; however, by looking at this controller, it is obvious that the discrete error, $e_d(k)$, obtained at time $t = k\delta$ is used to update the states of the controller at time $t = (k+1)\delta$ and that the control law (13) does not take into account the measurements already available at time $t = k\delta + \tau$ for the required input calculations. That is, controller (11)-(13) induces a time delay to the original system while using the available information. Fig. 1 depicts the schematic procedure of the control law calculations and updates. To deal with this additional delay, a discrete observer may be used in the proposed controller structure for the required input calculations. The discrete observer is presented in Appendix A.

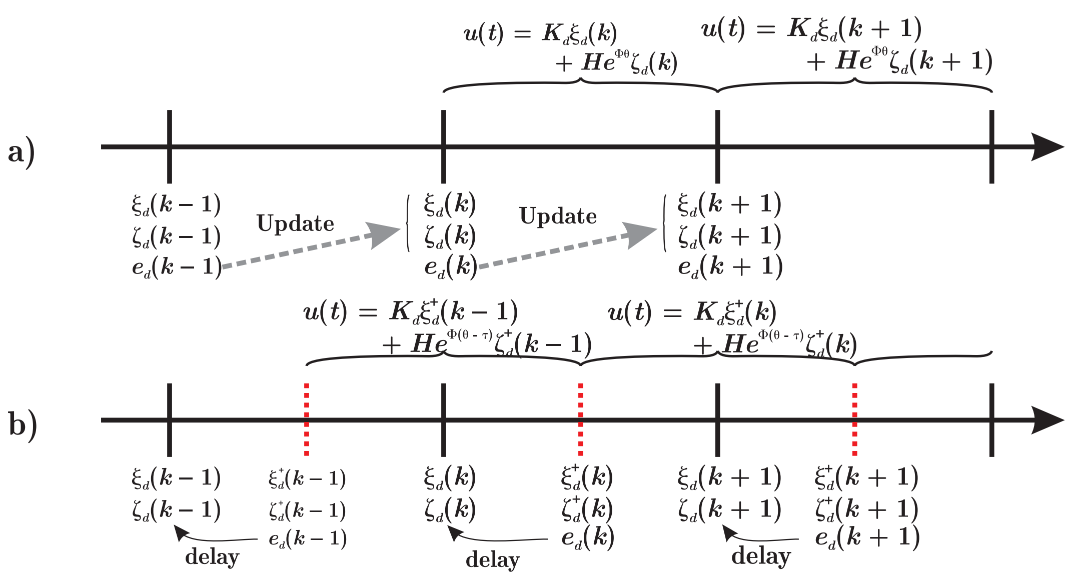

Fig. 1: Procedure of the control law calculations and updates. a) Typical update procedure. b) Proposed controller update procedure.

If one considers the deviation of the steady state $\underline{x}(t) = x(t) - \pi(w(t), \lambda)$, using the central manifold theory, it is possible to find a suitable mathematical model of the system given by

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) - B_0Hz(t) + B_0u(t) + f_2\left(\underline{x}(t), w(t), \lambda\right) \quad (14)$$

$$\dot{z}(t) = \Phi z(t) \quad (15)$$

$$e(t) = C_0\underline{x}(t-\tau) + h_2\left(\underline{x}(t-\tau), w(t-\tau), \lambda\right) \quad (16)$$

where

$$A_0 = \left[\frac{\partial f}{\partial x}\right]_{(0,0,0)}, \ B_0 = \left[\frac{\partial f}{\partial u}\right]_{(0,0,0)} \text{ and } C_0 = \left[\frac{\partial h}{\partial x}\right]_{(0,0)},$$

while f_2 and h_2 contain the second or higher order terms, which vanish at the origin along with their first order derivatives. $z \in \mathbb{R}^r$ represents an immersion similar to (7). As seen, (14)-(16) is an extended representation of (1) because of immersion (15), that allows the calculation of the steady state input $\gamma(w, \lambda)$.

Let us now recall that the discrete output is sampled with a period δ at time $t = k\delta$, for $k = 0, 1, 2, \ldots$, but it is available after a delay $\tau = \varepsilon\delta$ at time $t_k = (k + \varepsilon)\delta$, i.e.

$$e(t_k) = h(x(k\delta), w(k\delta), \lambda) =: e_d(k)$$

Thus, at time t_k it is possible to estimate $\underline{x}(k\delta) = \underline{x}(t_k - \tau)$ and $z(k\delta) = z(t_k - \tau)$ by using $e_d(k)$ in a discrete observer with the structure (A.4)-(A.5) (see Appendix A), which requires a discretized version of the linear approximation of (14)-(16) from one sampling instant to the next (i.e. from $t = k\delta$ to $t = (k+1)\delta$). Such a discretized version is given by

$$\underline{x}_d(k+1) = A_{d0}\underline{x}_d(k) - M_{d0}z_d(k) + \int_0^\delta e^{A_0\varsigma}B_0u(k\delta + \delta - \varsigma)\,d\varsigma \quad (17)$$

$$z_d(k+1) = \Phi_d z_d(k) \quad (18)$$

$$e_d(k) = C_0 x_d(k) \quad (19)$$

where $x_d(k) = x(k\delta)$, $z_d(k) = z(k\delta)$,

$$A_{d0} = e^{A_0\delta}, \ \Phi_d = e^{\Phi\delta} \text{ and } M_{d0} = \int_0^\delta e^{A_0\varsigma}B_0He^{\Phi(\delta-\varsigma)}d\varsigma. \quad (20)$$

Notice that in order to calculate the integration term in (17), one needs to know the behavior of the input from $t = k\delta$ to $t = (k+1)\delta$. This input and the discrete observer which update its states at each instant t_k, $k = 0, 1, 2, \ldots$, is presented in the following theorem.

Theorem 4. *Let us define the matrices*

$$B_{d0,1} = \int_0^{\delta-\tau} e^{A_0\varsigma}B_0d\varsigma, \quad M_{d0,1} = \int_0^{\delta-\tau} e^{A_0\varsigma}B_0He^{\Phi(\delta-\varsigma)}d\varsigma$$
$$B_{d0,2} = \int_{\delta-\tau}^{\delta} e^{A_0\varsigma}B_0d\varsigma, \quad M_{d0,2} = \int_{\delta-\tau}^{\delta} e^{A_0\varsigma}B_0He^{\Phi(\delta-\varsigma)}d\varsigma \quad (21)$$

$$\overline{A}_d = \begin{pmatrix} A_{d0} & -M_{d0} \\ 0 & \Phi_d \end{pmatrix}, \qquad \overline{C}_d = \begin{pmatrix} C_0 & 0 \end{pmatrix}, \quad (22)$$

where A_{d0}, Φ_d and M_{d0}, are the discretized nominal matrix of the linear approximation of system (1) described in (20). Assume conditions 1 and 2 hold. In addition, immersion (7)-(8) exists and so do the pairs $(A_{d0}, B_{d1,0})$ and $\left(\overline{A}_d, \overline{C}_d\right)$, which are stabilizable and observable, respectively. Then, a local solution for the Delayed Output Discrete Robust Regulation Problem is given by the following discrete controller

$$u(t) = K_d\xi_d^+(k) + He^{\Phi(\theta+\tau)}\zeta_d^+(k), \quad \theta \in [0, \delta), \ t = t_k + \theta, \quad (23)$$

$$\xi_d^+(k) = (I - G_{d1}C_0)\,\xi_d(k) + G_{d1}e_d(k), \quad t_k = (k + \varepsilon)\delta, \quad (24)$$

$$\zeta_d^+(k) = -G_{d2}C_0\xi_d(k) + \zeta_d(k) + G_{d2}e_d(k), \quad k = 0, 1, 2, \ldots \quad (25)$$

$$\xi_d(k+1) = (A_{d0} + B_{d0,1}K_d)\,\xi_d^+(k) - M_{d0,2}\zeta_d^+(k) + B_{d0,2}K_d\xi_d^+(k-1) + M_{d0,2}\Phi_d\zeta_d^+(k-1), \quad (26)$$

$$\zeta_d(k+1) = \Phi_d\zeta_d^+(k), \quad (27)$$

where K_d, and $G_d = \begin{pmatrix} G_{d1}^T & G_{d2}^T \end{pmatrix}^T$ render Schur the matrices $(A_{d0} + B_{d1,0}K_d)$ and $\left(\overline{A}_d - G_d\overline{C}_d\overline{A}_d\right)$, respectively, and given the existence of the symmetric matrices $P > 0$ and $Q > 0$, the Linear Matrix Inequalities (LMI)

$$P - \Psi_0^T P \Psi_0 \geq 2Q \quad (28)$$

$$\begin{pmatrix} Q & -\Psi_0^T P \Psi_1 \\ -\Psi_1^T P \Psi_0 & Q - \Psi_1^T P \Psi_1 \end{pmatrix} > 0 \quad (29)$$

hold, where

$$\Psi_0 = \begin{pmatrix} A_{d0} + B_{d0,1}K_d & -B_{d0,1}K_d & -M_{d0,1} \\ 0 & (I - G_{d1}C_0)A_{d0} & (G_{d1}C_0 - I)M_{d0} \\ 0 & -G_{d2}C_0A_{d0} & \Phi_d + G_{d2}C_0M_{d0} \end{pmatrix} \quad (30)$$

$$\Psi_1 = \begin{pmatrix} B_{d0,2}K_d & -B_{d0,2}K_d & -M_{d0,2}\Phi_d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (31)$$

Proof. See Appendix C. □

Remark 5. Notice that eq. (17) represents the dynamic estimation of the error $x - \pi(w, \lambda)$, while (27) is, actually, the estimation of the immersion. Eqs.

(24) and (25) are used to calculate the controller state updates from the delayed output while the input (23) is composed of two terms, the first one is a feedback error while the second one contains the exponential holder which allows to generate a continuous input which guarantees the zero error even at the intersampling period. Notice also that, according to the proof of theorem 4, the convergence rate of the closed loop system can be estimated with the eigenvalues of the matrix Q and the matrix in inequality (29).

An interesting case of controller (23)-(25) is when $\Phi = 0$ in immersion (7)-(8). In this case, the input (23) remains constant between sampling instants and is given by

$$u(t) = K_d \xi_d^+(k) + H\zeta_d^+(k), \qquad t = (k+\varepsilon)\delta + \theta,\ \theta \in [0, \delta), \tag{32}$$

while the dynamics of controller (24)-(27) takes the form

$$\begin{aligned}
\xi_d(k^+) &= (I - G_{d1}C_0)\,\xi_d(k) + G_{d1}e_d(k)\\
&\quad k = 0, 1, 2, \ldots\\
\zeta_d(k^+) &= -G_{d2}C_0\xi_d(k) + \zeta_d(k) + G_{d2}e_d(k) \qquad (33)\\
\xi_d(k+1) &= A_{d0}\xi_d(k^+) - B_{d0}\zeta_d(k^+) + B_{d0,1}u(k^+)\\
&\quad + B_{d0,2}u(k^+ - 1)\\
\zeta_d(k+1) &= \zeta_d(k^+)
\end{aligned}$$

where $B_{d0} = B_{d0,1} + B_{d0,2}$.

In the next section the performance and robustness of the proposed control scheme is evaluated for the tracking of constant and time-varing references in two study cases.

3 Study case

3.1 Disturbances rejection in a biological reactor used for wastewater treatment

Anaerobic digestion (AD) is a biological process in which, a consortium of anaerobic microorganisms transforms the organic matter into a biogas that it is composed of a mixture of methane (CH_4) and carbon dioxide (CO_2), as the main components, and di-hydrogen, carbon monoxide and di-hydrogen sulphur asthe secondary components (Moletta, 2005). This bioprocess is widespread in natural environment and is commonly used to treat industrial wastewater because of the advantages over other available technologies: a) it produces very low amounts of sludge (compared to aerobic digestion) and valuable energy (methane) from the reduction of the chemical oxygen demand (COD) present in the wastewater; b) it is well adapted to concentrated wastes such as agricultural and food industry wastewaters (Batstone et al., 2000; Antonelli et al., 2003) and c) it is able to operate under severe operating conditions, i.e. high-strength effluents and short hydraulic retention times. However, its industrial implementation has been limited, because of the difficulties involved in achieving the efficient operation of this bioprocess which have a reputation of being unstable and difficult-to-control with intrinsic characteristics that constitute major obstacles of the application of this technology such as: severe nonlinearities, load disturbances, system uncertainties, constraints on manipulated and state variables and limited on-line measurement information. The control problem is further hindered when trying to meet stringent environmental regulations of key process variables, such as the chemical oxygen demand (COD), whose measurements are difficult, time consuming and/or plagued with large time delays associated to laboratory analysis, transportation lags (such as flow through pipes), measurement sensors (measurement delays), and control actuators (manipulated input delays) that may be critical in some control actions (Cushing, 1977; Christofides and El-Farra, 2005). This particular problem has been addressed by proposing a number of control techniques, ranging from classical PI or PID controllers to more advanced control schemes (Aguilar et al., 2004; Flores-Estrella et al., 2016); however, most of these works have been focused on the development of theoretical control approaches and the performances have not been validated experimentally. Adaptive controllers have shown to be well fitted for their implementation in pilot scale (Renard et al., 1988; Monroy et al., 1996) and real-life-scale (Polihronakis et al., 1993).

The proposed model-based robust discrete regulation scheme case fully meets closed-loop objectives such as tracking, regulation and disturbance attenuation. This scheme is designed to regulate the organic matter concentration measured as chemical oxygen demand (COD) in AD processes by using the dilution rate as the manipulated variable and delayed COD measurements.

The bioprocess model

A relatively simple mathematical model may be derived from the mass balances of the species involved in the acidogenic phase of the AD process.

This model is described by the following ordinary differential equations (Bernard et al., 2001):

$$\dot{X} = (\mu - \alpha D)\, X \quad (34)$$

$$\dot{S} = (S_{in} - S)\, D - \frac{\mu X}{Y} \quad (35)$$

where X, S and S_{in} are respectively, the concentrations of acidogenic bacteria, COD, and inlet COD. Parameter α ($0 \leq \alpha \leq 1$) denotes the biomass fraction that is retained by the reactor bed, i.e., $\alpha = 0$ for the ideal fixed-bed bioreactor and $\alpha = 1$ for the ideal continuous stirred tank reactor. Y is the biomass yield coefficient for COD degradation. $D = D(t) \geq 0$ denotes the dilution rate and it is supposed to be bounded, i.e., $D^- \leq D(t) \leq D^+$. The specific growth rate is given by the nonlinear Monod equation in which most parameters are badly or inadequately known (Van-Impe et al., 1998; Dochain and Vanrolleghem, 2001):

$$\mu = \frac{\mu_{\max} S}{K_S + S}$$

where $\mu_{\max}$ and K_S are the maximum specific growth rate and the half saturation parameter associated with S, respectively. For normal operating conditions, the biomass always exists (i.e. $X > 0$) and there is (always) substrate consumption in the bioreactor, that is, $S - S_{in} > 0$, which is assumed positive definite for practical operation. Despite the simplicity of model (34)-(35), it does exhibit some of the key properties which render anaerobic digesters difficult to operate and control. In this particular study case, the proposed controller was implemented under two different situations that are common in practice: constant and periodic disturbance rejection under uncertain scenarios.

Constant disturbances rejection

It is well known that wastewater treatment processes are subject to unpredictable changes on the influent concentrations, due mainly to the quality of the supplied raw materials and by-products. These load changes introduce serious control problems in wastewater treatment (WWT) plants that must be compensated for the plant proper operation.

In this case, the design of the proposed controller is based on the nominal values of parameters, influent and set-point COD concentrations, which are assumed constant. These lead to the following linear exosystem

$$\dot{w} = 0 \qquad w \in \mathbb{R}^+$$
$$S_{in} = w_1$$
$$S_r = w_2$$

which describes the constant disturbance and set-point. By defining the error $e = S - Sr = S - w_2$, the solution of the FIB eqs. (4)-(5) is

$$X_{ss} = \frac{X_0}{\alpha (w_1 - w_2)^{-1} Y^{-1} X_0 \left(1 - e^{-\mu(w_1)t}\right) + e^{-\mu t}} =: \pi_1(w, \lambda)$$
$$S_{ss} = w_2 =: \pi_2(w, \lambda)$$
$$D_{ss} = \frac{\mu(w_1)}{Y(w_1 - w_2)} \pi_2(w, \lambda) =: \gamma(w, \lambda)$$

where the subscript ss represents the steady state values. Then, it is clear that for large enough time, both $X_{ss} = \alpha^{-1} Y(w_1 - w_2)$ and $D_{ss} = \alpha^{-1} \mu(w_1)$ are constant. In this case, by using immersion (7) with $\Phi = 0$ and $z \in \mathbb{R}$, it is possible to devise a controller of the form (32)-(33). For simulation purposes, we used the nominal parameters listed in Table 1. Then,with a sampling time equal to $\delta = 1/3$d, and a measurement delay equal to $\tau = 0.3\delta$, using discrete LQR techniques, we calculated the gains $K_d = (0.16, -0.36)$, $G_{d1} = (-0.14, 0.69)^T$ and $G_{d2} = -0.18$.

To resemble actual WWT plant conditions of influent substrate concentrations, we introduced changes on S_{in}. At $t = 0$, S_{in} was set to $21\text{kg} \cdot \text{m}^{-3}$ (110% higher than the nominal value) and changed to 12, 8 and $16\text{kg} \cdot \text{m}^{-3}$ at days 20, 40 and 50, respectively. At the beginning of the simulation run, the parameter variation percentage from the nominal values of $\mu_{\max}$, K_S, α and Y were -15%, -20%, -10% and 10%, respectively (see Table 1 for those parameters variations), while the reference signal was set to $3\text{kg} \cdot \text{m}^{-3}$. As seen in Fig. 2a, the proposed nonlinear discrete regulator was able to track the predetermined set-point trajectory in the face of the load and parametric changes. Furthermore, the controller coped with the drastic set-point change (induced at day 30) and the 60% disturbance on S_{in} at day 50. Fig. 2b depicts the time evolution of the manipulated variable, $D(t)$, which exhibited rare excursions to saturation.

Table 1: Nominal parameters and variations for WWT process.

Parameter	Nominal value	Percentage of variation from the nominal value				
		$0 \leq t \leq 20$	$20 < t \leq 30$	$30 < t \leq 40$	$40 < t \leq 50$	$t > 50$
$\mu_{\max}\left(\mathrm{d}^{-1}\right)$	1.25	−15%	25%	25%	25%	25%
$K_S\left(\mathrm{kg}\cdot\mathrm{m}^{-3}\right)$	4.96	−20%	−20%	25%	25%	25%
α	0.5	−10%	−10%	15%	15%	15%
Y	1/6.6	10%	10%	10%	−15%	−15%
$S_{in}\left(\mathrm{kg}\cdot\mathrm{m}^{-3}\right)$	10	110%	20%	20%	−20%	60%

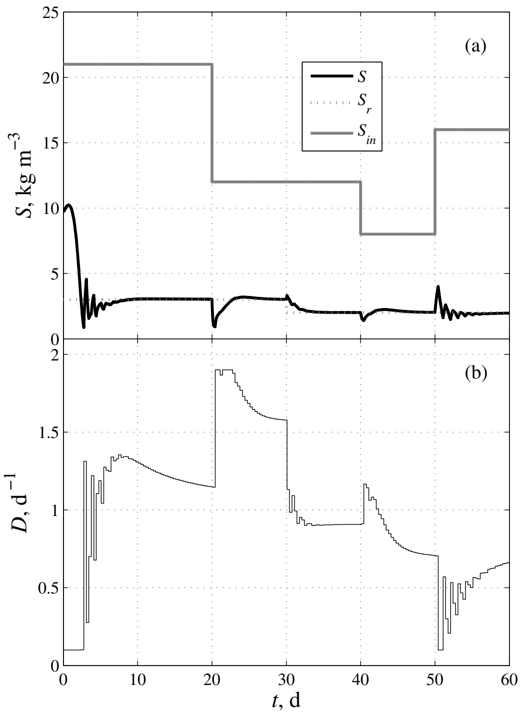

Fig. 2: Constant disturbance rejection simulation results for an anaerobic digestion process. (a) Substrate concentration: (—) reactor concentration; (- · -) influent concentration and (- - -) reference. (b) Dilution rate.

Periodic disturbances rejection

Among the different challenges raised by the control of the biological processes, one is specific to some classes of processes like municipal WWT plants for which the organic load is typically unknown and time varying, yet with a typical periodicity of one day that follows the human domestic activities and the related production of wastewater with a "nominal" periodic value which is roughly known. For this reason, we consider the regulation of the COD concentration, S, around a predetermined set-point, S_r, under the influence of persistent periodic disturbances on the influent COD concentration which may be given by

$$S_{in} = a + b\sin\left(\frac{2\pi}{T}t + \phi\right)$$

where T is the time period and a, b and ϕ are unknown parameters. These disturbances are taken into account in the following linear exosystem

$$\begin{aligned} \dot{w} &= s(w) \\ S_{in} &= w_1 \\ S_r &= w_4 \end{aligned}$$

where

$$s(w) = \frac{2\pi}{T}\begin{pmatrix} w_2 & w_3 & -w_2 & 0 \end{pmatrix}^T$$

Clearly, the error is $e = S - Sr = S - w_4$ which is zero when $S = w_4 =: \pi_2(w)$. Then, it is straightforward to show that the steady state control input and the steady state biomass are

$$X_{ss} = \frac{X_0}{e^{-\mu(w_4)t} + \alpha\mu(w_4)Y^{-1}X_0\int_0^t (w_1 - w_4)^{-1} e^{-\mu(w_4)(t-\varsigma)}d\varsigma} =: \pi_1(w,\lambda)$$

$$D_{ss} = \frac{\mu(w_4)}{Y(w_1 - w_4)}X_{ss} =: \gamma(w,\lambda)$$

where X_0 denotes the initial biomass concentration. Clearly, for relatively long times, X_{ss} becomes a periodic function and for small oscillations in S_{in} ($a \gg b$), and so does D_{ss} which results in

$$D_{ss} \approx \frac{\mu(w_4)}{\alpha}\left[1 + A\sin\left(\frac{2\pi}{T}t + \beta\right)\right]$$

where A and β are some constants. In this case, the immersion for $\gamma(w,\mu) = D_{ss}$ takes the linear form of (7)-(8) with

$$\Phi = \frac{2\pi}{T}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ and } H = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

that yields the following exponential holder

$$\Phi_d = e^{\Phi\delta} = \begin{pmatrix} 1 & \sin\left(\frac{2\pi}{T}\delta\right) & 1-\cos\left(\frac{2\pi}{T}\delta\right) \\ 0 & \cos\left(\frac{2\pi}{T}\delta\right) & \sin\left(\frac{2\pi}{T}\delta\right) \\ 0 & -\sin\left(\frac{2\pi}{T}\delta\right) & \cos\left(\frac{2\pi}{T}\delta\right) \end{pmatrix},$$

$$He^{\Phi(\theta-\tau)} = \begin{pmatrix} 1 \\ \sin c\,(\theta-\tau) \\ 1-\cos c\,(\theta-\tau) \end{pmatrix}^T.$$

Thus, one can design a robust regulator to reject perturbations using the proposed scheme. For simulation purposes, we used an oscillation period $T = 1$d for the input COD concentration with a sampling time $\delta = 1/3$d, a delay $\tau = \delta/2$ and the same nominal parameters and load disturbances for S_{in} used in the previous simulation example (see Table 1). In addition, some changes in the amplitude and the phase angle, b and ϕ, where also induced in the simulation run. Using discrete LQR procedures, we calculated the gains $K_d = (0.21, -0.14)$, $G_{d1} = (-0.14, 0.72)^T$ and $G_{d2} = (-0.20, 0.09, 0.06)^T$, which fulfill the conditions of Theorem 4.

Fig. 3 depicts the response of the proposed robust regulator in the face of persistent load disturbances. As seen, the effluent COD concentration exhibited a deviation at the start-up of the bioreactor due to the saturation of the control input (D), the erroneous initial conditions and changes on the nominal values of parameters (see Fig. 3b); however, the effect of these circumstances was compensated by the controller after 10 days and the effluent COD concentration was strongly regulated despite the persistent periodic disturbances of the inlet COD concentration. As seen in Fig. 3a, the system output satisfactorily tracked the desired trajectory under the influence of parameter uncertainties, set-point changes and periodic load disturbances.

3.2 Experimental implementation

In order to test the proposed robust regulator in an actual highly uncertain real scenario, its experimental implementation was carried out in a up-flow fixed-bed digestor for 70 days with an effective volume of $V = 2.8$l (see Fig. 4) by using raw tequila vinasses collected from a tequila factory located at La Laja-Jalisco, Mexico, with approximately 34g COD/l as a substrate feed composition. The pH of the inflow tequila vinasses was regulated between 6.5 and 7.0 by adding a NaOH solution and a remotely controllable peristaltic pump was used to ensure the desired influent flow rate.

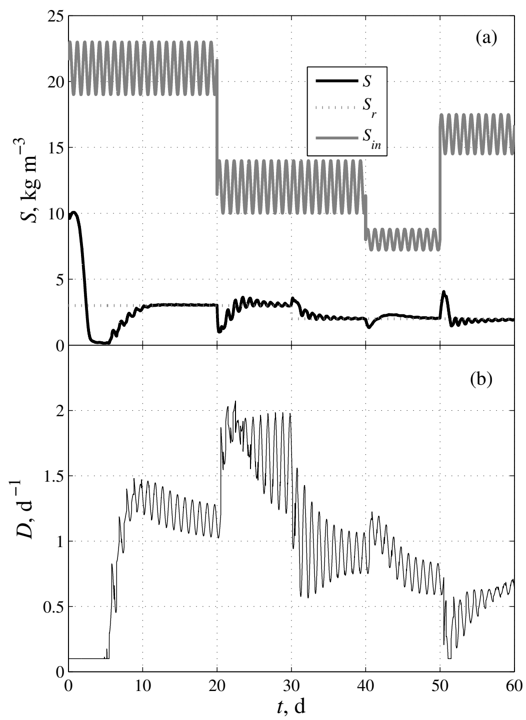

Fig. 3: Periodic disturbance rejection simulation results for an anaerobic digestion process. (a) Substrate concentration: (—) reactor concentration; (-· -) influent concentration and (- - -) reference. (b) Dilution rate.

To guarantee homogeneous conditions, fresh substrate was mixed with the recycled liquid just before entering the reactor. Bioreactor temperature was also regulated at 35±1°C. A National Instruments cRIO9004 device equipped with analogical and digital cards was used in the acquisition, treatment and storage of the on-line measured variables, such as pH, temperature, pressure and the biogas and wastewater flow rates. In addition, off-line measurements of soluble COD, partial, total and intermediate alkalinity, bicarbonate and volatile fatty acids were also performed during experiments; however, due to the scope of this contribution and space limitations, only the input and output COD readings are presented in this work. The soluble COD was determined by the closed reflux colorimetric method by using the HACH digester DBR200 and the spectrophotometer DR2800, which are usually used in most tequila distillery quality control labs. The regulation problem was considered as a constant disturbances rejection case similar to that previously described by using a sampling rate of 1 day and a sampling processing delay of 5h, which is the time needed to determine the COD concentration of the sample by the HACH methodology (i.e., $\delta = 1$ day and $\tau = 5$h) while the dilution rate was used as the manipulated variable. Since the parameters for Tequila vinasses were not identified, the nominal parameters used to design the controller were those shown in Table 1 which correspond to typical values reported in the literature for AD processes of wine vinasses (i.e. the use of these values introduces additional model uncertainties to further test the performance of the proposed robust regulator).

The ability of the proposed robust regulator to handle AD process disturbances was examined by introducing abrupt variations on the inflow COD concentration. Initial inflow vinasses were diluted at approximately 60% and then, at day 29, raw vinasses were fed to the digester causing a severe load disturbance (see Fig. 5a). Set-point changes were also induced in the experimental run. The set-point trajectory was initially predetermined at 1.5g COD/l for 15 days, increased to 2.5 and kept at this value for 30 additional days and then set back to 1.5 for the rest of the experimental run.

Fig. 5b depicts the resulting COD removal efficiency, calculated as the removed COD concentration with respect to the inflow COD (i.e. $100\,(S_{in} - S)\,/S_{in}$) whereas Fig. 5c depicts the control effort given by the time evolution of the manipulated variable, D^{-1}, the residence time (inverse dilution rate). The proposed robust regulator successfully maintained the desired set-point trajectory despite the severe load disturbance. A slight overshoot was observed in the COD removal efficiency profile following the introduction of the load. The robust regulator damped the behavior after 4 days without producing any detrimental transient. The regulator outputs were well behaved, requiring no excessive control action (see Fig. 5c). On the other hand, the robust regulator handled the servo problem quite well. As expected, the initial COD removal efficiency was below the predetermined set-point value, which was achieved shortly after two days once the control effort compensated the parameter uncertainties and erroneous initial conditions. Moreover, the control effort attained the new values within 24 hours after its respective change and in the face of the severe load disturbance.

Conclusions

A robust discrete output tracking and disturbance rejection scheme for a class of nonlinear processes with output delays was proposed in this paper. It is composed of a discrete error feedback controller, a linear discrete estimator of the steady state input and an exponential holder which allows to reproduce the exact steady state input needed to avoid the intersampling ripple. This robust regulator was successfully tested in a simulation and actual experimental study case under different scenarios including parameter uncertainty, set-point changes and external disturbances.

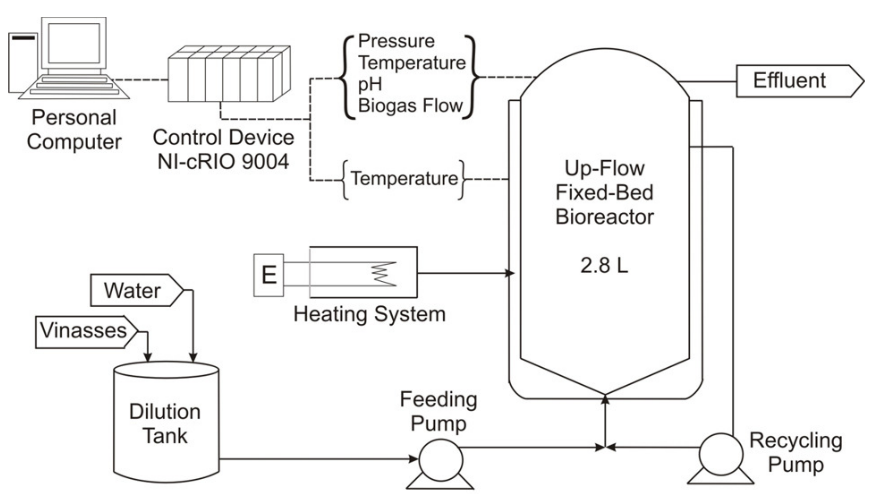

Fig. 4: Schematic view of the AD process used in the experimental run of example 2.

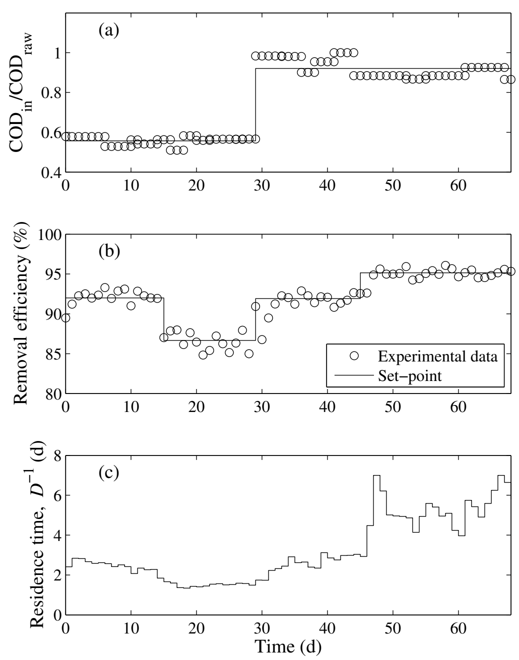

Fig. 5: Experimental application results of the robust regulator to an anaerobic digestion process. (a) **Disturbance:** Dimensionless inflow COD concentration. (b) **Output:** Removal efficiency. (c) **Input:** Resident time.

The proposed structure showed excellent tracking and load rejection capabilities in the presence of significant modeling errors and parametric uncertainties.

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Appendix

A Discrete observer with instantaneous update

Let us consider the linear system

$$\dot{x}(t) = Fx(t) \tag{A.1}$$

with a discrete output $y_d(k\delta) = Cx(k\delta)$, where $k = 0, 1, 2, \ldots$ and a sampling period equal to δ.

The discrete realization of system (A.1) is

$$x_d(k+1) = F_d x_d(k) \tag{A.2}$$

where $x_d(k) = x(k\delta)$ and $F_d = e^{F\delta}$. A typical discrete observer for system (A.2) is

$$\xi_d(k+1) = (F_d - G_d C)\,\xi_d(k) + G_d y_d(k) \tag{A.3}$$

where $F_d - G_d C$ must be Schur in order to guarantee the convergence of such observer. However, by looking at the discrete observer (A.3), it is evident that the discrete output obtained at time $t = k\delta$ is used to update the observer states at time $t = (k+1)\,\delta$inducing an additional delay in the use of the available information which may be taken into account in the calculation of the output feedback control law. For this purpose let us propose the following observer:

$$\begin{aligned} \xi_d(k^+) &= (I - G_d C)\,\xi_d(k) + G_d y_d(k) && \text{(A.4)} \\ \xi_d(k+1) &= F_d \xi_d(k^+) && \text{(A.5)} \end{aligned}$$

where $\xi_d(k^+)$, calculated with eq. (A.4), represents the update of the observer states at time $t = k\delta$ using the output obtained at the same instant, while $\xi_d(k+1)$, computed in eq. (A.5), represents the expected behavior at $t = (k+1)\delta$ just prior to the updating step control scheme with the new measurement. In this case, when the observer is used in an output feedback controller, it is better to use $\xi_d(k^+)$ instead of $\xi_d(k)$, since $\xi_d(k^+)$ has already been corrected with newly available information. Sufficient conditions to guarantee the convergence of such observer (A.4)-(A.5) are stated in the following lemma.

Lemma 6. *Let us consider system (A.1) with a discrete output $y_d(k) = Cx(k)$, where $k = 0, 1, 2, \ldots$ and a sampling period equal to δ. If the pair (F_d, C) is detectable, using matrix G_d such that $F_d - G_dCF_d$ is Schur, observer (A.4)-(A.5) guarantees that* $\lim_{k\to\infty}[x_d(k) - \xi_d(k^+)] = 0$

Proof. Let us define the observer error

$$e_d = x_d - \xi_d. \tag{A.6}$$

Since $x_d(k) = x_d(k^+)$, the dynamics of e_d is

$$\begin{aligned} e_d(k^+) &= (I - G_dC)\, e_d(k) && \text{(A.7)} \\ e_d(k+1) &= F_d e_d(k^+) && \text{(A.8)} \end{aligned}$$

By considering the updated version (i.e. $k^+ + 1$) in (A.7) and replacing (A.8), one gets

$$e_d(k^+ + 1) = (I - G_dC)\, e_d(k+1) = (I - G_dC)\, F_d e_d(k^+)$$

and thus, if the pair (F_d, CF_d) is observable. Then, a matrix G_d may be calculated such that $F_d - G_dCF_d$ is Schur and the error $e_d(k^+)$ will converge to zero (i.e. $\lim_{k\to\infty}[x_d(k) - \xi_d(k^+)] = 0$). To proof that the pair (F_d, CF_d) is observable if the pair (F_d, C) is observable, let us consider its observability matrix

$$O = \begin{pmatrix} CF_d \\ CF_d^2 \\ \vdots \\ CF_d^n \end{pmatrix},$$

where $F_d \in \mathbb{R}^{n\times n}$, then by using the Hamilton-Cailey theorem (Kailath, 1980)

$$F_d^n = a_0 I + a_1 F_d + \cdots + a_{n-1} F_d^{n-1},$$

and the observability matrix becomes

$$O = \begin{pmatrix} CF_d \\ CF_d^2 \\ \vdots \\ a_0 C + a_1 CF_d + \cdots + a_{n-1} CF_d^{n-1} \end{pmatrix}.$$

Since F_d is obtained through a discretization of matrix F then $a_0 \neq 0$ and O has full rank if the pair (F_d, C) is observable. □

B Stability of discrete delayed systems

Since the delay in the output of system (1) induces a delay in the structure of the closed loop system (C.3), we are interested in the stability of systems with the form

$$x(k+1) = \Psi_0 x(k) + \Psi_1 x(k-1) \tag{B.1}$$

where $\vartheta \in \mathbb{R}^n$. The following Lemma states sufficient conditions to guarantee the stability of such a system.

Lemma 7. *The discrete linear system (B.1) is asymptotically stable if there exists the positive definite matrices $P > 0$ and $Q > 0$ such that the LMI's*

$$\begin{aligned} P - \Psi_0^T P \Psi_0 &\geq 2Q && \text{(B.2)} \\ \begin{pmatrix} Q & -\Psi_0^T P \Psi_1 \\ -\Psi_1^T P \Psi_0 & Q - \Psi_1^T P \Psi_1 \end{pmatrix} &\geq 0 && \text{(B.3)} \end{aligned}$$

hold.

Proof. Let define the Lyapunov function

$$V = x^T(k)\, Px(k) + x^T(k-1)\, Qx(k-1)$$

with the increase of V given by

$$\begin{aligned} \Delta V =& x^T(k+1)\, Px(k+1) - x^T(k)(P - Q)\, x(k) \\ &- x^T(k-1)\, Qx(k-1) \\ =& x^T(k)\left(\Psi_0^T P \Psi_0 - P + Q\right) x(k) + x^T(k-1)\, \Psi_1^T P \Psi_0 x(k) \\ &+ x^T(k)\, \Psi_0^T P \Psi_1 x(k-1) + x^T(k-1)\, \Psi_1^T P \Psi_1 x(k-1). \end{aligned}$$

If (B.2) holds, then

$$\Delta V \leq -\begin{pmatrix} x(k) \\ x(k-1) - x(k) \end{pmatrix}^T \begin{pmatrix} Q & -\Psi_0^T P \Psi_1 \\ -\Psi_1^T P \Psi_0 & Q - \Psi_1^T P \Psi_1 \end{pmatrix} \begin{pmatrix} x(k) \\ x(k-1) - x(k) \end{pmatrix}$$

and if (B.3) is satisfied, $\Delta V < 0$, and as a consequence, system (B.1) is asymptotically stable, concluding the proof. □

C Proof of Theorem 4

Proof. Eqs. (24) and (25) are used to calculate the updates ξ_d and ζ_d at a time $t_k = (k+\varepsilon)\delta$ when the output is available; thus, for $k\delta \leq t < (k+1)\delta$, the input (23) is

$$u(t) = \begin{cases} K_d\xi_d^+(k-1) + He^{\Phi(t-k\delta+\delta)}\zeta_d^+(k-1) \\ \text{if } k\delta \leq t < k\delta+\tau \\ K_d\xi_d^+(k) + He^{\Phi(t-k\delta)}\zeta_d^+(k) \\ \text{if } k\delta+\tau \leq t < (k+1)\delta \end{cases},$$

because at time $k\delta < t < t_k$ the discrete controller state has not yet been updated. Replacing this input, the integral in (17) becomes

$$\int_0^\delta e^{A_0\varsigma}B_0 u(k\delta+\delta-\varsigma)\,d\varsigma = B_{d0,1}K_d\xi_d^+(k) + M_{d0,1}\zeta_d^+(k) + B_{d0,2}K_d\xi_d^+(k-1) + M_{d0,2}e^{\Phi\delta}\zeta_d^+(k-1)$$

where $B_{d0,1}$, $B_{d0,2}$, $M_{d0,1}$ and $M_{d0,2}$ are defined in (21), that yields the following discrete linear approximation of system (14)-(16) by using input (23)

$$\begin{aligned} \underline{x}_d(k+1) =& A_{d0}\underline{x}_d(k) - M_{d0}z_d(k) + B_{d0,1}K_d\xi_d^+(k) \\ &+ M_{d0,1}\zeta_d^+(k) + B_{d0,2}K_d\xi_d^+(k-1) \\ &+ M_{d0,2}e^{\Phi\delta}\zeta_d^+(k-1), \qquad \text{(C.1)} \\ z_d(k+1) =& \Phi_d z_d(k). \end{aligned}$$

By defining

$$\vartheta_d(k) = \begin{pmatrix} \underline{\xi}_d(k) \\ \underline{\zeta}_d(k) \end{pmatrix} = \begin{pmatrix} \underline{x}_d(k) - \xi_d(k) \\ z_d(k) - \zeta_d(k) \end{pmatrix}$$

and since $x_d^+(k) = x_d(k)$ and $z_d^+(k) = z_d(k)$, the dynamics of ϑ_d can be written as

$$\begin{aligned} \vartheta_d^+(k) &= \left(I - G_d\overline{C}_d\right)\vartheta_d(k) \\ \vartheta_d(k+1) &= \overline{A}_d\vartheta_d^+(k) \end{aligned} \qquad \text{(C.2)}$$

where $\overline{A}_d$ and $\overline{C}_d$ are defined in (22). Then, as described in Lemma 6 of Appendix A, if G_d renders Schur the matrix $\left(\overline{A}_d - G_d\overline{C}_d\overline{A}_d\right)$ then $\lim_{k\to\infty}\vartheta_d^+(k) = 0$, which implies that the states $\xi_d^+(k)$ and $\zeta_d^+(k)$ of the proposed controller converge to $\underline{x}_d(k)$ and $z_d(k)$, respectively.

To demonstrate that the stability condition is fulfilled, let us consider the case when $z = 0$, then by defining $\underline{\vartheta}_d^+(k) = \text{col}\left\{\underline{x}_d^+(k), \underline{\xi}_d^+(k), \underline{\zeta}_d^+(k)\right\}$, from (C.1) and (C.2) it holds that

$$\underline{\vartheta}_d^+(k+1) = \Psi_0\underline{\vartheta}_d^+(k) + \Psi_1\underline{\vartheta}_d^+(k-1) \qquad \text{(C.3)}$$

where Ψ_0 and Ψ_1 are described in (30)-(31). Notice that since K_d and G_d render Schur the matrices $(A_{d0} + B_{d1,0}K_d)$ and $\left(\overline{A}_d - G_d\overline{C}_d\overline{A}_d\right)$, respectively, then Ψ_0 is also Schur and by using the result of Lemma 7 of Appendix B, if the LMIs (28)-(29) hold, then system (C.3) is asymptotically stable. For the regulation part given by condition *2.1*, it follows from the FIB eqs. (4)-(5) that when ξ_d goes asymptotically to zero, the dynamics of ζ_d tends to the discrete version of the immersion dynamics (7)-(8) and after using the exponential holder, the resulting input (23) is

$$He^{\Phi(\theta+\tau)}\zeta_d^+(k) = Hz(t) = \gamma(w,\lambda),$$

which makes invariant the zero tracking error submanifold (Castillo-Toledo and Di'Gennaro, 2002). □